Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SCHUBERT CALCULUS OF A COXETER GROUP
Autor: Hiller, Howard L.
Kapitel: 5. H_w AS A AND PARABOLICS
DOI: https://doi.org/10.5169/seals-51740

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 05.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Example. Recall that in H_{Σ_3} we computed

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) \ .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

5. H_W AS A W-MODULE AND PARABOLICS

If (W, S) is a Coxeter system and $\theta \subseteq S$ then (W_θ, θ) is also a Coxeter system [6, p. 20] and W_θ is called a *parabolic* subgroup of W. In addition, it is easy to see that a geometric realization (Δ, Σ) of (W, S) can be restricted to a geometric realization of (W_θ, θ). The collection $\{W_\theta\}_{\theta \subset S}$ of parabolic forms a lattice of $2^{|S|}$ distinct subgroups where, for example, $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$. We will eventually be concerned with the set of left cosets of W_θ in W. We define $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$. The following basic result is well-known [6, p. 37 and p. 45].

THEOREM 5.1. *Every element w of W can be uniquely expressed as $w^\theta \cdot w_\theta$ with $w^\theta \in W^\theta$, $w_\theta \in W_\theta$ and furthermore $l(w) = l(w^\theta) + l(w_\theta)$.*

This immediately yields

COROLLARY 5.2. *W^θ is a complete set of left coset representations for W_θ in W and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in H_W. The most straightforward approach is to compute exactly the action of W on H_W. This is easily done by exploiting the computation (4.1).

THEOREM 5.3. *The structure of H_W as a W-module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \sum\limits_{\substack{\gamma \\ ws_\alpha \to w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$

Proof. As in (4.5), choose A such that $\varepsilon \Delta_g(A) = \delta_{gw}$. Then, since c is a W-map

$$\begin{aligned} s_\alpha X_w &= c(s_\alpha A) = \sum_{w' \in W} \varepsilon \Delta_{w'}(s_\alpha A) X_{w'} \\ &= \sum_{w' \in W} \varepsilon \Delta_{w'}(1 - \alpha^* \Delta_\alpha)(A) X_{w'} \\ &= X_w - \sum_{w' \neq w} (\varepsilon \Delta_{w'} \alpha^*) \Delta_\alpha(A) X_{w'} \\ &= X_w - \sum_{\substack{\gamma \\ g \to w'}} (g^{-1}(\gamma)^v, \alpha) \varepsilon \Delta_{gs_\alpha}(A) X_{w'} \qquad \text{by (4.1)} \\ &= \sum_{\substack{\gamma \\ g \to w' \\ l(gs_\alpha) = l(g)+1 \\ gs_\alpha = w}} (g^{-1}(\gamma)^v, \alpha) X_{w'} \\ &= X_w - \sum_{\substack{\gamma \\ ws_\alpha \to w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} \end{aligned}$$

Note, that the summation in the next to the last line is non-vacuous if and only if $l(ws_\alpha) = l(w) - 1$. This completes the proof.

COROLLARY 5.4. *$X_w \in H_W^{W_\theta}$ if $w \in W^\theta$.*

Proof. Immediate from (5.3) and the definition of W^θ.

The following elementary result shows that the X_w, $w \in W^\theta$, are actually an $\mathbf{R}$-basis for $H_W^{W_\theta}$.

LEMMA. *If a finite group G acts on a real vector space V via the regular representation and H is a subgroup of G, then*

$$\dim_{\mathbf{R}}(V^H) = [G : H].$$

Proof. Let $\{e_g\}_{gG}$ be a basis for V, so that

$$g' \cdot e_g = e_{gg'}$$

Then if $\chi = \sum_{g \in G} \xi_g \in V^H$, we claim $\xi_g = \xi_{g'}$, if $g \equiv g' \pmod{H}$. Indeed, if $g = g'h$, $h \in H$, then

$$\begin{aligned} \xi_{g'} &= \text{coefficient of } e_{g'} \chi \text{ in} \\ &= \text{coefficient of } e_{g'} \text{ in } h^{-1}\chi \cdot \\ &= \text{coefficient of } e_{g'h} \text{ in } \chi \\ &= \xi_g. \end{aligned}$$

Hence, there are at most $[G : H]$ free parameters in determining $\chi \in V^H$ and clearly each choice gives an invariant. This finishes the proof.

COROLLARY 5.6. *dim* $(H_{W^\theta}^W) = [W : W_\theta] = |W^\theta|$ *and the* X_w, $w \in W^\theta$, *are an* **R**-*basis for* $H_{W^\theta}^W$.

Proof. Chevalley [8] has shown that S_W, hence H_W, is abstractly equivalent to the regular representation of W, as a W-module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for H_W to $H_{W^\theta}^W$. We have

THEOREM 5.7. *If* $w, w' \in W^\theta$ *and in* H_W

$$X_w \cdot X_{w'} = \sum_{w'' \in W} c(w, w', w'') X_{w''}$$

then in $H_{W^\theta}^W$

$$X_w \cdot X_{w'} = \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''}$$

Proof. One need only observe that the vector space map $r : H_W \to H_{W^\theta}^W$ given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying r to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of W_θ-invariants. Notice that an appropriate Giambelli formula for $H_{W^\theta}^W$ is not as easily obtained. This is because the Giambelli formula for H_W gives X_w as a polynomial in the X_{s_α}'s and not all of these are in the algebra $H_{W^\theta}^W$, so this is not quite the right thing.

6. APPLICATION: THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair (W, W_θ) of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra $H_{W^\theta}^W$. In this section, we examine the particular case $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$ where Σ_m denotes the symmetric group on m letters. Indeed, Σ_{n+k} is the Weyl group of the root system of type A_{n+k-1}, which we recall briefly here. Let $V' = \mathbf{R}^{n+k}$