Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SCHUBERT CALCULUS OF A COXETER GROUP
Autor: Hiller, Howard L.
Kapitel: 4. Pieri formula
DOI: https://doi.org/10.5169/seals-51740

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where ρ is the sum of the fundamental weights. Hence one can attempt to compute the operators Δ_w on ρ^N.

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that $\rho^N \notin I_W$, maybe even for all ρ in the interior of the fundamental chamber.

4. Pieri formula

Recall that the algebra of operators $\mathbb{\Delta}_W$ was generated by both the Δ_α's and the multiplication operators ω^*. Using the basis constructed in (2.9), if one composes such operators, say $\omega^* \circ \Delta_w$ or $\Delta_w \circ \omega^*$, it should be possible to express them linearly in terms of the operators Δ_g, $g \in W$. Of course, our eventual concern is with the algebra $\mathbb{\Delta}_W$ and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0 .$$

So, if we compute the commutator $[\Delta_w, \omega^*]$ a quick application of ε will yield a formula for $\varepsilon \cdot \Delta_w \circ \omega^*$. Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator $w^{-1}\Delta_w$ (recall $W \subset \mathbb{\Delta}_W$). The main result is

Theorem 4.1. *If* $w \in W, \omega \in V^*$, *then in* End $S(V)$,

$$[w^{-1}\Delta_w, \omega^*] = \sum_{w' \xrightarrow{\gamma} w} (w'^{-1}(\gamma)^v, \omega)\, w^{-1}\Delta_{w'} .$$

We will now fix a reduced decomposition $w = s_{\alpha_1}, \ldots, s_{\alpha_k}$ and write s_i for s_{α_i} and $w_i = s_{\alpha_n} \ldots s_{\alpha_i}$. First we have the following easy observation.

Lemma 4.2. *Let* $\theta_i = s_k \ldots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i)$, $1 \leqslant i \leqslant k$. *Then*

(i) $w^{-1}\Delta_w = \Delta_{\theta_1}\Delta_{\theta_2}\ldots\Delta_{\theta_k}$

and

(ii) $s_{\theta_i}(w_i^{\wedge})^{-1} = w^{-1}$

Proof. Note by (2.2 ii, iv) $s_\alpha \Delta_\alpha = \Delta_\alpha$. Hence

$$w^{-1}\Delta_w = s_k \dots s_1 \Delta_{\alpha_1} \dots \Delta_{\alpha_k} = \Delta_{s_k \dots s_1(\alpha_1)} s_k \dots s_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$
$$= \Delta_{\theta_1} w_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$

and induction completes the argument. The second remark follows precisely as in (1.3).

Proof of (4.1). We compute

$$[w^{-1}\Delta_w, \omega^*] = [\Delta_{\theta_1} \circ \dots \circ \Delta_{\theta_k}, \omega^*]$$
$$= \sum_{j=1}^{k} \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \dots \Delta_{\theta_k} \quad .$$

Let us call the j-th summand P_j. Firstly, observe that $[\Delta_{\theta_j}, \omega^*] = (\theta_j^v, \omega)\, s_{\theta_j}$ by (2.2 vii). If we substitute this into P_j and drag the reflection s_{θ_j} over to the left we get

$$P_j = \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega) \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} s_{\theta_j} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_v^j, \omega)\, s_{\theta_j} \Delta_{s_{\theta_j}(\theta_1)} \dots \Delta_{s_{\theta_j}(\theta_{j-1})} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega)\, s_{\theta_j} (w\hat{j})^{-1} \Delta_{w\hat{j}} \quad .$$

To see this final identity we must argue, by (4.2), that $s_{\theta_j}(\theta_i) = \pm\, \theta_{i,\hat{j}}$ where $\theta_{i,\hat{j}} = s_k \dots \hat{s_j} \dots s_{i+1}(\alpha_i)$. (As in the above remark, $\theta_{i,\hat{j}}$ is the θ_i for $w\hat{j} = s_1 \dots \hat{s_j} \dots s_k$.) But, we can assume $i < j$, in which case

$$s_{\theta_j}(\theta_i) = s_k \dots s_{j+1} s_j s_{j+1} \dots s_k \big(s_k \dots s_j s_{j+1} \dots s_{i+1}(\alpha_i)\big)$$
$$= s_k \dots \hat{s_j} \dots s_{i+1}(\alpha_i) = \theta_{i,\hat{j}} \quad .$$

And now, by (4.2 ii)

$$P_j = (\theta_j^v, \omega)\, w^{-1} \Delta_{w\hat{j}}$$

and $s_{w\hat{j}(\theta_j)}(w\hat{j}) = w$, so $w\hat{j} \xrightarrow{w\hat{j}(\theta_j)} w$. Finally, (1.2) allows us to reindex by the immediate subwords of w

$$\sum_{j=1}^{k} P_j = \sum_{\substack{\gamma \\ w' \to w}} \big((w')^{-1}(\gamma)^v, \omega\big)\, w^{-1} \Delta_{w'}$$

and the proof is complete.

COROLLARY 4.3.

$$\Delta_w \cdot \omega^* = w \cdot \omega^* \cdot w^{-1} \Delta_w + \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)|, \omega(\Delta_{w'} .$$

Proof. Multiply (4.1) by w.

COROLLARY 4.4.

$$\varepsilon \cdot \Delta_w \cdot \omega^* = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma^v), \omega) \varepsilon \cdot \Delta_{w'} .$$

Proof. The first term in the right-hand side of (4.3) is annihilated by ε. It is now easy to dualize the above and obtain

THEOREM 4.5 (Pieri formula). *If* $w \in W, \alpha \in \Sigma$, *then in* H_W

$$X_{s_\alpha} \cdot X_w = \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, w_\alpha) X_{w'} .$$

Proof. Choose A such that $\varepsilon \cdot \Delta_{w'}(A) = \delta_{ww'}$, for example $\sigma(X_w)$. Then, by (3.4 ii)

$$\begin{aligned} X_{s_\alpha} \cdot X_w &= c(\omega_\alpha A) \\ &= \sum_{w' \in W} \varepsilon \Delta_{w'}(\omega_\alpha A) X_{w'} \\ &= \sum_{w' \in W} \varepsilon \cdot \Delta_{w'} \omega_\alpha^*(A) X_{w'} \\ &= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha) \varepsilon \circ \Delta_g(A) \Big) X_{w'} \\ &= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha) \delta_{gw} \Big) X_{w'} \\ &= \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, \omega_\alpha) X_{w'} . \end{aligned}$$

Of course, it is also possible to rewrite this formula in the following equivalent form.

COROLLARY 4.6.

$$X_{s_\alpha} \cdot X_w = \sum_{\substack{\beta \in \Delta + \\ l(ws_\beta) = l(w) + 1}} (\beta^v, \omega_\alpha) X_{ws_\beta} .$$

Proof. It suffices to note $\sigma_\gamma w = w'$ if and only if $w \sigma_{w^{-1}(\gamma)} = w'$.

Example. Recall that in H_{Σ_3} we computed

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) \ .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

5. H_W AS A W-MODULE AND PARABOLICS

If (W, S) is a Coxeter system and $\theta \subseteq S$ then (W_θ, θ) is also a Coxeter system [6, p. 20] and W_θ is called a *parabolic* subgroup of W. In addition, it is easy to see that a geometric realization (Δ, Σ) of (W, S) can be restricted to a geometric realization of (W_θ, θ). The collection $\{W_\theta\}_{\theta \subset S}$ of parabolic forms a lattice of $2^{|S|}$ distinct subgroups where, for example, $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$. We will eventually be concerned with the set of left cosets of W_θ in W. We define $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$. The following basic result is well-known [6, p. 37 and p. 45].

THEOREM 5.1. *Every element w of W can be uniquely expressed as $w^\theta \cdot w_\theta$ with $w^\theta \in W^\theta$, $w_\theta \in W_\theta$ and furthermore $l(w) = l(w^\theta) + l(w_\theta)$.*

This immediately yields

COROLLARY 5.2. *W^θ is a complete set of left coset representations for W_θ in W and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in H_W. The most straightforward approach is to compute exactly the action of W on H_W. This is easily done by exploiting the computation (4.1).

THEOREM 5.3. *The structure of H_W as a W-module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \sum\limits_{\substack{\gamma \\ ws_\alpha \to w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$