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**Autor:** Hiller, Howard L.  
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This result rapidly yields our version of the “basis theorem” of the Schubert calculus, namely

**THEOREM 2.9.**  $\text{Ker } (c) = I_W$  and  $c$  induces an isomorphism  $S_W \approx H_W$ .

*Proof.* For the first assertion, by (2.8), it suffices to compute

$$\begin{aligned} c(\lambda d) &= \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0} \\ &= \lambda |W| X_{w_0}. \end{aligned}$$

Finally,  $c$  is clearly onto by construction.

In the next section we will work on producing an explicit section for  $c$ .

*Remark.* Demazure’s proof, though restricted to Weyl groups, is done integrally. In that situation,  $c$  is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when  $W$  preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

### 3. GIAMBELLI FORMULA

We begin with an easy lemma.

**LEMMA 3.1.**  $\Delta$  is quasi-multiplicative, i.e.

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The first clause is immediate since the conditions implies that reduced decompositions of  $w$  and  $w'$  can be juxtaposed to yield a reduced decomposition of  $ww'$ . Now suppose  $w = s_\alpha w'$  and  $l(s_\alpha w') = l(w') - 1$  (that this is the only possibility that follows from (1.1)). Then  $w' = s_\alpha(s_\alpha w')$  and

$$l(w') = 1 + (l(w') - 1) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part  $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$ . But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on  $l(w)$  completes the proof.

COROLLARY 3.2.  $\varepsilon \cdot \Delta_{w'} \Delta_{w^{-1}w_0} = \delta_{ww'} \Delta_{w_0}$  on  $S_N(V)$ .

*Proof.* If  $w' = w$ , then by (1.4) and (3.1)

$$\Delta_w \Delta_{w^{-1}w_0} = \Delta_{w_0}$$

and the result follows.

We now need only consider  $w' \neq w$ , but with  $l(w) = l(w')$ , (otherwise, we are done for dimensional reasons). Thus

$$l(w') + l(w^{-1}w_0) = l(w') + (l(w_0) - l(w)) = l(w_0)$$

and

$$l(w'w^{-1}w_0) = l(w_0) - l(w'w^{-1}) \neq l(w_0)$$

So by (3.1),  $\Delta_{w'} \Delta_{w^{-1}w_0} = 0$ , and the proof is complete.

It is now easy to dualize this to the following assertion:

COROLLARY 3.3 (Giambelli formula).  $c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) = X_w$ . Hence in particular,  $c\left(\frac{d}{|W|}\right) = X_{w_0}$ .

$$\begin{aligned} \text{Proof. } c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) &= \sum_{w' \in W} \varepsilon \Delta_{w'} \left( \Delta_{w^{-1}w_0} \left( \frac{d}{|W|} \right) \right) X_{w'} \\ &= \sum_{\substack{w' \in W \\ l(w') = l(w)}} \delta_{ww'} \varepsilon \Delta_{w_0} \left( \frac{d}{|W|} \right) X_{w'} \\ &= X_w \end{aligned} \quad \text{by (2.5).}$$

Note that the map  $\sigma : X_w \mapsto \Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)$  is a vector space section for  $c$ . In the remainder of this section we will find other  $I_W$ -equivalent expressions for  $X_{w_0}$  and use these to put  $\sigma$  in a more manageable form. We will call  $X_{w_0}$  the *fundamental class* of the cohomology ring  $H_W$ .

*Example.* Let  $W = W(A_{n-1}) = \Sigma_n$ . As usual, the positive roots  $\Delta^+$  are  $\{e_i - e_j : i < j\}$  where  $\{e_i\}$  is the standard basis of  $\mathbb{R}^n$ . Hence, the fundamental class is  $c$  of a multiple of the Vandermonde determinant, namely

$$\frac{1}{n!} \begin{vmatrix} 1 & e_n & e_n^2 & \dots & e_n^{n-1} \\ 1 & e_{n-1} & e_{n-1}^2 & \dots & e_{n-1}^{n-1} \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ 1 & e_1 & e_1^2 & \dots & e_1^{n-1} \end{vmatrix}$$

In this example we used the standard basis for  $V$ . The following result indicates that a Coxeter generalization of the fundamental weight basis is more appropriate in our situation. Recall the *fundamental weights*  $\{\omega_\alpha\}_{\alpha \in \Sigma}$  are given by the requirement

$$(\omega_\alpha, \beta^\vee) = \delta_{\alpha\beta}$$

We now have

LEMMA 3.4.

- (i)  $\Delta_\beta(\omega_\alpha) = \delta_{\alpha\beta}$ ,
- (ii)  $c(\omega_\alpha) = X_{s_\alpha}$ ,
- (iii)  $c(\alpha) = \sum_{\beta \in \Sigma} (\alpha, \beta^\vee) X_{s_\beta}$ .

*Proof.*

- (i)  $\Delta_\beta(\omega_\alpha) = \beta^{-1}(\omega_\alpha - s_\beta(\omega_\alpha)) = \beta^{-1}(\omega_\alpha - (\omega_\alpha - (\omega_\alpha, \beta^\vee)\beta)) = (\omega_\alpha, \beta^\vee) = \delta_{\alpha\beta}$
- (ii)  $c(\omega_\alpha) = \sum_{w \in W} \varepsilon \Delta_w(\omega_\alpha) X_w = \sum_{\beta \in \Sigma} \Delta_\beta(\omega_\alpha) X_{s_\beta} = X_{s_\alpha}$
- (iii) Since  $\alpha = \sum_{\beta \in \Sigma} (\alpha, \beta^\vee) \omega_\beta$ , the result follows immediately from (ii).

This result tells us that if we can write  $X_w$  as  $c$  of some polynomial in the  $\{\omega_\alpha\}_{\alpha \in \Sigma}$  or  $\{\alpha\}_{\alpha \in \Sigma}$  we will have also written  $X_w$  as a polynomial in the  $X_{s_\alpha}$ 's. We will often abbreviate the Cartan matrix entries by  $c_{\alpha, \beta} = (\alpha, \beta^\vee) = -\frac{\|\alpha\|}{\|\beta\|} \cos\left(\frac{\pi}{m_{\alpha\beta}}\right)$ . In practice, it is maximally efficient to write  $X_w$  as a polynomial in the simple roots, since then an easy substitution will yield either a polynomial in the weights or a polynomial in the original coordinate variables  $e_1, \dots, e_n$ .

It is possible to relate the fundamental class  $X_{w_0}$ , with the invariant theory of  $W$ .

PROPOSITION 3.5. *Let  $f_1, \dots, f_n$  be fundamental invariants for  $W$ . Then, if  $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)$  is the Jacobian of these polynomials there is a real number  $\lambda$  such that*

$$c(\lambda J) = X_{w_0}.$$

*Proof.* This follows from the stronger, well-known assertion that  $d$  divides  $J$  [20, p. 85]. (It also follows from the theory of complete intersection rings.)

In the interest of understanding the Giambelli formula (3.3) we deduce some formulae for  $\Delta_w(d)$ . If  $\{\alpha_i\}_{1 \leq i \leq n}$  are distinct positive roots we denote by  $d_{\alpha_1, \dots, \alpha_n}$  the product  $d \cdot \prod_{i=1}^n \alpha_i^{-1} = \prod_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_i}} \alpha$ . It is easy to see

LEMMA 3.6.

$$s_\beta(d_{\alpha_1, \dots, \alpha_n}) = \begin{cases} d_{s_\beta(\alpha_1), \dots, s_\beta(\alpha_n), \beta} & \text{if } \beta = \alpha_j, \\ -d_{s_\beta(\alpha_1), \dots, s_\beta(\alpha_n)} & \text{otherwise.} \end{cases}$$

*Proof.* Since  $s_\beta$  permutes the set  $\Delta^+ - \{\beta\}$ , it also permutes

$$\Delta^+ - \{\beta, \alpha_1, \dots, \alpha_n, s_\beta(\alpha_1), \dots, s_\beta(\alpha_n)\},$$

where  $\beta \neq \alpha_i$ , for all  $i$ . Hence

$$\begin{aligned} s_\beta(d_{\alpha_1, \dots, \alpha_n}) &= s_\beta(d_{\beta, \alpha_1, \dots, \alpha_n, s_\beta(\alpha_1), \dots, s_\beta(\alpha_n)}) \cdot s_\beta(\beta) \cdot s_\beta^2(\alpha_1) \circ \dots \circ s_\beta^2(\alpha_n) \\ &= d_{\beta, \alpha_1, \dots, \alpha_n, s_\beta(\alpha_1), \dots, s_\beta(\alpha_n)} \cdot (-\beta) \cdot \alpha_1 \cdot \dots \cdot \alpha_n \\ &= -d_{s_\beta(\alpha_1), \dots, s_\beta(\alpha_n)} \end{aligned}$$

Similarly in the other case.

PROPOSITION 3.7.

$$\Delta_\beta(d_{\alpha_1, \dots, \alpha_n}) = \begin{cases} \sum_{\substack{s \neq \phi \\ s \subseteq \{1, \dots, \hat{j}, \dots, n\}}} (-1)^{|s|} \prod_{i \in s} c_{\alpha_i, \beta} & \\ \beta^{|s|-1} d_{\{\alpha_i : i \in s\}, \widehat{s_\beta(\alpha_1), \dots, s_\beta(\alpha_j), \dots, s_\beta(\alpha_n)}} & \text{if } \beta = \alpha_j \\ d_{\alpha_1, \dots, \alpha_n, \beta} + d_{s_\beta(\alpha_1), \dots, s_\beta(\alpha_n), \beta} & \text{otherwise} \end{cases}$$

*Proof.* The second case is easy so we look at the first

$$\begin{aligned} \Delta_\beta(d_{\alpha_1, \dots, \alpha_n}) &= \beta^{-1} (d_{\alpha_1, \dots, \alpha_n} - d_{s_\beta(\alpha_1), \dots, \widehat{s_\beta(\alpha_j)}, \dots, s_\beta(\alpha_n), \beta}) \\ &= \beta^{-1} [d_{\alpha_1, \dots, \alpha_n, \widehat{s_\beta(\alpha_1)}, \dots, \widehat{s_\beta(\alpha_j)}, \dots, s_\beta(\alpha_n)} \\ &\quad \cdot (s_\beta(\alpha_1) \cdot \dots \cdot \widehat{s_\beta(\alpha_j)} \cdot \dots \cdot s_\beta(\alpha_n) - \alpha_1 \cdot \dots \cdot \widehat{\alpha_j} \cdot \dots \cdot \alpha_n)] \\ &= d_{\alpha_1, \dots, \alpha_n, s_\beta(\alpha_j), \dots, s_\beta(\alpha_j), \dots, s_{\beta n}(\alpha)} \\ &\quad \beta^{-1} \left( \prod_{i \neq j} (\alpha_j - (\alpha_j, \beta^v) \beta) - \prod_{i \neq j} \alpha_i \right) \end{aligned}$$

and after writing the product as a sum the desired expression follows.

It is possible to use (3.7) to explicitly compute polynomial expressions for  $X_w$ .

*Example.* Let  $W = W(A_2)$  where  $A_2$  is the root system in  $\mathbf{R}^3$  with simple roots  $\Sigma = \{\alpha = e_1 - e_2, \beta = e_2 - e_3\}$  and the additional positive root  $\alpha + \beta = e_1 - e_3$ . Hence  $X_{w_0} = \frac{1}{6} \alpha \beta (\alpha + \beta)$ . As a check of this we compute the Jacobian  $J$  of the fundamental invariant. Recall

$$\sigma_1 = -(e_2 + e_3)(e_2 + e_3) + e_2 e_3$$

and

$$\sigma_2 = -(e_2 + e_3) e_2 e_3,$$

where we have eliminated  $e_1 = -(e_2 + e_3)$ . Then:

$$J = 3(e_2^2 e_3 - e_3^2 e_2) + 2(e_2^3 - e_3^3) = d,$$

so also,  $X_{w_0} = \frac{1}{6} J$ . Now by (3.7) we can compute

$$\Delta_\alpha \left( \frac{d}{6} \right) = \frac{1}{6} (2d_\alpha) = \frac{1}{3} \beta (\alpha + \beta)$$

and

$$\Delta_\beta \Delta_\alpha \left( \frac{d}{6} \right) = \frac{1}{3} (\Delta_\beta d_\alpha) = \frac{1}{3} (d_{\alpha, \beta} + d_{s_\beta(\alpha), \beta}) = \frac{1}{3} (\alpha + \beta + \alpha) = \frac{1}{3} (2\alpha + \beta)$$

so that:

$$X_{s_\alpha s_\beta} = \frac{1}{3} \beta (\alpha + \beta) \quad \text{and} \quad X_{s_\alpha} = \frac{1}{3} (2\alpha + \beta) = \omega_\alpha$$

as one easily checks.

Now since the Cartan matrix is  $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  we have

$$\alpha = 2\omega_\alpha - \omega_\beta$$

$$\beta = -\omega_\alpha + 2\omega_\beta$$

so for example

$$\begin{aligned} X_{s_\alpha s_\alpha} &= \frac{1}{3} (-X_{s_\alpha} + 2X_{s_\alpha})(X_{s_\alpha} + X_{s_\beta}) \\ &= \frac{1}{3} (-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) \end{aligned}$$

which will be confirmed further in the next section.

*Remark.* In the crystallographic case, it follows from the Weyl denominator formula (see [6, p. 185], [2, p. 17]) that

$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where  $\rho$  is the sum of the fundamental weights. Hence one can attempt to compute the operators  $\Delta_w$  on  $\rho^N$ .

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that  $\rho^N \notin I_W$ , maybe even for all  $\rho$  in the interior of the fundamental chamber.

#### 4. PIERI FORMULA

Recall that the algebra of operators  $\Delta_W$  was generated by both the  $\Delta_\alpha$ 's and the multiplication operators  $\omega^*$ . Using the basis constructed in (2.9), if one composes such operators, say  $\omega^* \circ \Delta_w$  or  $\Delta_w \circ \omega^*$ , it should be possible to express them linearly in terms of the operators  $\Delta_g$ ,  $g \in W$ . Of course, our eventual concern is with the algebra  $\Delta_W$  and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0.$$

So, if we compute the commutator  $[\Delta_w, \omega^*]$  a quick application of  $\varepsilon$  will yield a formula for  $\varepsilon \cdot \Delta_w \circ \omega^*$ . Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator  $w^{-1} \Delta_w$  (recall  $W \subset \Delta_W$ ). The main result is

**THEOREM 4.1.** *If  $w \in W$ ,  $\omega \in V^*$ , then in  $\text{End } S(V)$ ,*

$$[w^{-1} \Delta_w, \omega^*] = \sum_{\substack{\gamma \\ w' \rightarrow w}} (w'^{-1}(\gamma)^v, \omega) w^{-1} \Delta_{w'}.$$

We will now fix a reduced decomposition  $w = s_{\alpha_1} \dots s_{\alpha_k}$  and write  $s_i$  for  $s_{\alpha_i}$  and  $w_i = s_{\alpha_n} \dots s_{\alpha_i}$ . First we have the following easy observation.

**LEMMA 4.2.** *Let  $\theta_i = s_k \dots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i)$ ,  $1 \leq i \leq k$ . Then*

$$(i) \quad w^{-1} \Delta_w = \Delta_{\theta_1} \Delta_{\theta_2} \dots \Delta_{\theta_k}$$

and

$$(ii) \quad s_{\theta_i}(w_i^\wedge)^{-1} = w^{-1}$$

*Proof.* Note by (2.2 ii, iv)  $s_\alpha \Delta_\alpha = \Delta_\alpha$ . Hence