

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SCHUBERT CALCULUS OF A COXETER GROUP
Autor: Hiller, Howard L.
Kapitel: 2. Demazure's basis theorem
DOI: <https://doi.org/10.5169/seals-51740>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta^+} \gamma \in S_N(V)$. The corresponding "anti-averaging" operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

2. DEMAZURE'S BASIS THEOREM

Let $\varepsilon : S(V) \rightarrow S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1} (1 - s_\alpha)$ as an $S(V)^W$ -endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

LEMMA 2.2. *If $w \in W, \alpha \in \Delta, u, v \in S(V)$ then*

- (i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)},$
- (ii) $\Delta_\alpha^2 = 0,$
- (iii) $s_\alpha = 1 - \alpha \Delta_\alpha,$
- (iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ (where the superscript denotes invariants)
- (v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u) \Delta_\alpha(v),$
- (vi) $\Delta_\alpha(I_W) \subset I_W,$
- (vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha,$

where ω^* denotes the operator multiplication by ω .

We now define Δ_W to be the subalgebra of the algebra of endomorphisms $\text{End}(S(V))$ generated by the Δ_α 's ($\alpha \in \Delta$) and $\omega^*, \omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \Delta_W$ by (2.2 iii).

There is a map $\varepsilon_* : \text{End } S(V) \rightarrow S(V)^*$ obtained by composition with ε and we write $\overline{\Delta}_W = \varepsilon_* \Delta_W \subseteq S(V)^*$. Double duality over \mathbf{R} gives a map

$$c : S(V) \xrightarrow{\delta} S(V)^{**} \xrightarrow{i^*} \overline{\Delta}_W^*.$$

We will write $H_W = \overline{\Delta}_W^*$, christened by Demazure, the cohomology ring of (Δ, Σ) and c the characteristic homomorphism. Demazure [11, Prop. 2] makes the basic observation that c induces a unique graded algebra and W -module structure on H_W compatible with $S(V)$. (We should mention here that H_W depends on the lengths of the simple roots though the notation obscures this.) The first task is to extend the class of operators $\Delta_\alpha = \Delta_{s_\alpha}$ from S to the entire Coxeter group W . Naturally, we will define $\Delta_w = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}$ where $w = s_{\alpha_1} \dots s_{\alpha_k}$ is a reduced decomposition of w , once we have proven that this definition is independent of the choice of the decomposition. Our information on Coxeter groups is a possible route but instead we follow Demazure's argument since it leads to worthwhile dividends. We begin with a few lemmas.

LEMMA 2.3: Let d denote $\prod_{\alpha \in \Delta^+} \alpha \in S_N(V)$. If $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$ is the longest word then

$$\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_N} = d^{-1}((-1)^N w_0 + \sum_{w \neq w_0} q_w w)$$

where $q_w \in \overline{S(V)}$.

Proof. We compute

$$\begin{aligned} \Delta_{\alpha_1} \dots \Delta_{\alpha_N} &= \alpha_1^{-1} (1 - s_{\alpha_1}) \dots \alpha_N^{-1} (1 - s_{\alpha_N}) \\ &= (-1)^N \alpha_1^{-1} s_1 \dots \alpha_N^{-1} s_N + \sum_{w \neq w_0} a(w) w \end{aligned}$$

where the index of summation in the last term is a consequence of (1.4). It now suffices to watch in the first term what happens to inverted roots α_i^{-1} as we pass the fundamental reflections s_i over to the right. Using (2.2 i) we get

$$(-1)^N \left(\prod_{i=1}^n s_{\alpha_1} \dots s_{\alpha_{i-1}} (\alpha_i) \right)^{-1} s_{\alpha_1} \dots s_{\alpha_N}$$

But by (1.3) this is $(-1)^N d^{-1} w_0$ since $w_0^{-1} = w_0$ converts all positive roots into negative roots by (1.4). We now let $q_w = da_w$ and the proof is complete.

LEMMA 2.4. If $f \in \text{End}(S(V))$ reduces the grading by N then $d \cdot f = \lambda J$ for some $\lambda \in \mathbf{R}$, where $J = \sum_{w \in W} (-1)^{l(w)} w$.

Proof. See [11, Prop. 1 (b)].

PROPOSITION 2.5: If $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$ as above, then $\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = d^{-1} J$.

Proof. By (2.4), $d\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = \lambda J = \lambda(-1)^N w_0 + \sum_{w \neq w_0} (-1)^{l(w)} \lambda w$.

Also by (2.3) $d\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = (-1)^N w_0 + \sum_{w \neq w_0} q_w w$. By Dedekind's theorem (see, e.g., [1]) the w 's are independent as automorphisms of $\overline{S(V)}$, so $\lambda = 1$ and the result follows.

We can now show

PROPOSITION 2.6. Δ_w is well defined.

Proof. By [6, IV § 1, Prop. 5], it suffices to show

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots$$

with $m_{\alpha\beta}$ terms on each side. But the dihedral root system $I_2(m_{\alpha\beta})$ or $A_1 \times A_1$ has $s_\alpha s_\beta s_\alpha \dots = s_\beta s_\alpha s_\beta \dots$ as its longest word and hence (2.5) completes the argument.

THEOREM 2.7. The $\{\Delta_w\}_{w \in W}$ are an $S(V)$ -basis for Δ_W and hence the $\{\varepsilon \circ \Delta_w\}_{w \in W}$ are an \mathbf{R} -basis for $\overline{\Delta}_W$.

Proof. By (2.2 v), it is easy to check the Δ_w 's generate Δ_W as an $S(V)$ -module. The linear independence follows from Dedekind's theorem, and the last statement is immediate.

We now define $\{X_w\}_{w \in W}$ to be the basis of $H_W = \overline{\Delta}_W^*$ dual to the basis $\{\varepsilon \cdot \Delta_w\}_{w \in W}$ of $\overline{\Delta}_W$, i.e.

$$X_w(\varepsilon \cdot \Delta_{w'}) = \delta_{ww'}$$

This immediately yields the following "coordinate-wise" description of c .

$$c(u) = \sum_{w \in W} \varepsilon \Delta_w(u) X_w.$$

First, we show c has the correct kernel. We need the following Lemma that follows from R. Steinberg [21].

LEMMA 2.8. If I is a graded ideal of $S(V)$ such that $I_W \subseteq I$ and $Rd \cap I = 0$, then $I = I_W$.

This result rapidly yields our version of the "basis theorem" of the Schubert calculus, namely

THEOREM 2.9. $\text{Ker}(c) = I_W$ and c induces an isomorphism $S_W \approx H_W$.

Proof. For the first assertion, by (2.8), it suffices to compute

$$\begin{aligned} c(\lambda d) &= \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0} \\ &= \lambda |W| X_{w_0}. \end{aligned}$$

Finally, c is clearly onto by construction.

In the next section we will work on producing an explicit section for c .

Remark. Demazure's proof, though restricted to Weyl groups, is done integrally. In that situation, c is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when W preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

3. GIAMBELLI FORMULA

We begin with an easy lemma.

LEMMA 3.1. Δ is quasi-multiplicative, i.e.

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first clause is immediate since the conditions implies that reduced decompositions of w and w' can be juxtaposed to yield a reduced decomposition of ww' . Now suppose $w = s_\alpha w'$ and $l(s_\alpha w') = l(w') - 1$ (that this is the only possibility that follows from (1.1)). Then $w' = s_\alpha(s_\alpha w')$ and

$$l(w') = 1 + (l(w') - 1) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$. But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on $l(w)$ completes the proof.