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Autor: Hiller, Howard L.

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There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta +} \gamma \in S_N(V)$. The corresponding "anti-averaging" operating is $\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u$.

2. Demazure's basis theorem

Let $\varepsilon : S(V) \to S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1}(1 - s_\alpha)$ as an $S(V)^W$-endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

Lemma 2.2. *If* $w \in W, \alpha \in \Delta, u, v \in S(V)$ *then*

(i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$,

(ii) $\Delta_\alpha^2 = 0$,

(iii) $s_\alpha = 1 - \alpha \Delta_\alpha$,

(iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ *(where the superscript denotes invariants)*

(v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$,

(vi) $\Delta_\alpha(I_W) \subset I_W$,

(vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha$,

where ω^* *denotes the operator multiplication by* ω.

We now define $\mathbb{A}_W$ to be the subalgebra of the algebra of endomorphisms $\operatorname{End}(S(V))$ generated by the Δ_α's ($\alpha \in \Delta$) and ω^*, $\omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \mathbb{A}_W$ by (2.2 iii).

There is a map $\varepsilon_* : \operatorname{End} S(V) \xrightarrow{i} S(V)^*$ obtained by composition with ε and we write $\overline{\triangle}_W = \varepsilon_* \triangle_W \subseteq S(V)^*$. Double duality over **R** gives a map

$$c : S(V) \xrightarrow{\delta} S(V)^{**} \xrightarrow{i^*} \overline{\triangle}{}^*_W .$$

We will write $H_W = \bar{\Delta}^*_W$, christened by Demazure, the cohomology ring of (Δ, Σ) and c the characteristic homomorphism. Demazure [11, Prop. 2] makes the basic observation that c induces a unique graded algebra and W-module structure on H_W compatible with $S(V)$. (We should mention here that H_W depends on the lengths of the simple roots though the notation obscures this.) The first task is to extend the class of operators $\Delta_\alpha = \Delta_{s_\alpha}$ from S to the entire Coxeter group W. Naturally, we will define $\Delta_w = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}$ where $w = s_{\alpha_1} \dots s_{\alpha_k}$ is a reduced decomposition of w, once we have proven that this definition is independent of the choice of the decomposition. Our information on Coxeter groups is a possible route but instead we follow Demazure's argument since it leads to worthwhile dividends. We begin with a few lemmas.

LEMMA 2.3: *Let d denote $\prod_{\alpha \in \Delta +} \alpha \in S_N(V)$. If $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$ is the longest word then*

$$\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_N} = d^{-1}((-1)^N w_0 + \sum_{w \neq w_0} q_w w)$$

where $q_w \in \overline{S(V)}$.

Proof. We compute

$$\begin{aligned} \Delta_{\alpha_1} \dots \Delta_{\alpha_N} &= \alpha_1^{-1}(1 - s_{\alpha_1}) \dots \alpha_N^{-1}(1 - s_{\alpha_N}) \\ &= (-1)^N \alpha_1^{-1} s_1 \dots \alpha_1^{-1} s_N + \sum_{w \neq w_0} a(w)\, w \end{aligned}$$

where the index of summation in the last term is a consequence of (1.4). It now suffices to watch in the first term what happens to inverted roots α_i^{-1} as we pass the fundamental reflections s_i over to the right. Using (2.2 i) we get

$$(-1)^N \Big(\prod_{i=1}^{n} s_{\alpha_1} \dots s_{\alpha_{i-1}}(\alpha_i) \Big)^{-1} s_{\alpha_1} \dots s_{\alpha_N}$$

But by (1.3) this is $(-1)^N d^{-1} w_0$ since $w_0^{-1} = w_0$ converts all positive roots into negative roots by (1.4). We now let $q_w = d a_w$ and the proof is complete.

LEMMA 2.4. *If* $f \in \mathrm{End}(S(V))$ *reduces the grading by* N *then* $d \cdot f = \lambda J$ *for some* $\lambda \in \mathbf{R}$, *where* $J = \sum_{w \in W} (-1)^{l(w)} w$.

Proof. See [11, Prop. 1 (b)].

PROPOSITION 2.5: *If* $w_0 = s_{\alpha 1} \dots s_{\alpha N}$ *as above, then* $\Delta_{\alpha 1} \dots \Delta_{\alpha N} = d^{-1} J$.

Proof. By (2.4), $d\Delta_{\alpha_1} \dots \Delta_{\alpha N} = \lambda J = \lambda(-1)^N w_0 + \sum_{w \neq w_0} (-1)^{l(w)} \lambda w$. Also by (2.3) $d\Delta_{\alpha_1} \dots \Delta_{\alpha N} = (-1)^N w_0 + \sum_{w \neq w_0} q_w w$. By Dedekind's theorem (see, e.g., [1]) the w's are independent as automorphisms of $\overline{S(V)}$, so $\lambda = 1$ and the result follows.

We can now show

PROPOSITION 2.6. Δ_w *is well defined.*

Proof. By [6, IV § 1, Prop. 5], it suffices to show

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots$$

with $m_{\alpha\beta}$ terms on each side. But the dihedral root system $I_2(m_{\alpha\beta})$ or $A_1 \times A_1$ has $s_\alpha s_\beta s_\alpha \dots = s_\beta s_\alpha s_\beta \dots$ as its longest word and hence (2.5) completes the argument.

THEOREM 2.7. *The* $\{\Delta_w\}_{w \in W}$ *are an* $S(V)$-*basis for* $\mathbb{A}_W$ *and hence the* $\{\varepsilon \circ \Delta_w\}_{w \in W}$ *are an* $\mathbf{R}$-*basis for* $\overline{\mathbb{A}}_W$.

Proof. By (2.2 v), it is easy to check the Δ_w'x generate $\mathbb{A}_W$ as an $S(V)$-module. The linear independence follows from Dedekind's theorem, and the last statement is immediate.

We now define $\{X_w\}_{w \in W}$ to be the basis of $H_W = \overline{\mathbb{A}}^*_W$ dual to the basis $\{\varepsilon \cdot \Delta_w\}_{w \in W}$ of $\overline{\mathbb{A}}_W$, i.e.

$$X_w(\varepsilon \cdot \Delta_{w'}) = \delta_{ww'}$$

This immediately yields the following "coordinate-wise" description of c.

$$c(u) = \sum_{w \in W} \varepsilon \Delta_w(u) X_w \ .$$

First, we show c has the correct kernel. We need the following Lemma that follows from R. Steinberg [21].

LEMMA 2.8. *If* I *is a graded ideal of* $S(V)$ *such that* $I_W \subseteq I$ *and* $\mathbf{R}d \cap I = 0$, *then* $I = I_W$.

This result rapidly yields our version of the "basis theorem" of the Schubert calculus, namely

THEOREM 2.9. Ker $(c) = I_W$ *and* c *induces an isomorphism* $S_W \approx H_W$.

Proof. For the first assertion, by (2.8), it suffices to compute

$$\begin{aligned} c(\lambda d) &= \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0} \\ &= \lambda \, | W | \, X_{w_0} . \end{aligned}$$

Finally, c is clearly onto by construction.

In the next section we will work on producing an explicit section for c.

Remark. Demazure's proof, though restricted to Weyl groups, is done integrally. In that situation, c is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when W preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

3. GIAMBELLI FORMULA

We begin with an easy lemma.

LEMMA 3.1. Δ *is quasi-multiplicative, i.e.*

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first clause is immediate since the conditions implies that reduced decompositions of w and w' can be juxtaposed to yield a reduced decomposition of ww'. Now suppose $w = s_\alpha w'$ and $l(s_\alpha w') = l(w') - 1$ (that this is the only possibility that follows from (1.1)). Then $w' = s_\alpha (s_\alpha w')$ and

$$l(w') = 1 + \big(l(w') - 1\big) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$. But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on $l(w)$ completes the proof.