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There is also a theory of anti-invariants, i.e. polynomials  $u \in S(V)$  such that  $w \cdot u = (-1)^{l(w)} u$ . The algebra of anti-invariants is written  $S(V)^{-W}$ . It is a free module of rank 1 over  $S(V)^W$  generated by the element  $d = \prod_{\gamma \in \Delta^+} \gamma \in S_N(V)$ . The corresponding "anti-averaging" operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

## 2. DEMAZURE'S BASIS THEOREM

Let  $\varepsilon : S(V) \rightarrow S_0(V) \approx \mathbf{R}$  denote the projection map. We begin by defining certain operators on  $S(V)$ , whose composition with  $\varepsilon$  should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let  $(W, S)$  be a Coxeter system and  $(\Delta, \Sigma)$  a geometric realization of it.

*Definition 2.1.* If  $\alpha \in \Delta$ , define  $\Delta_\alpha = \alpha^{-1} (1 - s_\alpha)$  as an  $S(V)^W$ -endomorphism of  $S(V)$ . (Note the division is legitimate since  $s_\alpha$  is the identity on the  $\ker(\alpha) = \alpha^\perp$ ; thinking of  $\alpha$  as a linear form  $x \mapsto (x, \alpha)$  in  $V^* = S_1(V)$ , of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

LEMMA 2.2. *If  $w \in W, \alpha \in \Delta, u, v \in S(V)$  then*

- (i)  $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)},$
- (ii)  $\Delta_\alpha^2 = 0,$
- (iii)  $s_\alpha = 1 - \alpha \Delta_\alpha,$
- (iv)  $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$  (where the superscript denotes invariants)
- (v)  $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u) \Delta_\alpha(v),$
- (vi)  $\Delta_\alpha(I_W) \subset I_W,$
- (vii)  $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha,$

where  $\omega^*$  denotes the operator multiplication by  $\omega$ .

We now define  $\Delta_W$  to be the subalgebra of the algebra of endomorphisms  $\text{End}(S(V))$  generated by the  $\Delta_\alpha$ 's ( $\alpha \in \Delta$ ) and  $\omega^*, \omega \in S(V)$ . Note  $\Delta_\alpha$  decreases the grading by  $(-1)$  and  $W \subseteq \Delta_W$  by (2.2 iii).

There is a map  $\varepsilon_* : \text{End } S(V) \rightarrow S(V)^*$  obtained by composition with  $\varepsilon$  and we write  $\overline{\Delta}_W = \varepsilon_* \Delta_W \subseteq S(V)^*$ . Double duality over  $\mathbf{R}$  gives a map

$$c : S(V) \xrightarrow{\delta} S(V)^{**} \xrightarrow{i^*} \overline{\Delta}_W^*.$$

We will write  $H_W = \overline{\Delta}_W^*$ , christened by Demazure, the cohomology ring of  $(\Delta, \Sigma)$  and  $c$  the characteristic homomorphism. Demazure [11, Prop. 2] makes the basic observation that  $c$  induces a unique graded algebra and  $W$ -module structure on  $H_W$  compatible with  $S(V)$ . (We should mention here that  $H_W$  depends on the lengths of the simple roots though the notation obscures this.) The first task is to extend the class of operators  $\Delta_\alpha = \Delta_{s_\alpha}$  from  $S$  to the entire Coxeter group  $W$ . Naturally, we will define  $\Delta_w = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}$  where  $w = s_{\alpha_1} \dots s_{\alpha_k}$  is a reduced decomposition of  $w$ , once we have proven that this definition is independent of the choice of the decomposition. Our information on Coxeter groups is a possible route but instead we follow Demazure's argument since it leads to worthwhile dividends. We begin with a few lemmas.

LEMMA 2.3: Let  $d$  denote  $\prod_{\alpha \in \Delta^+} \alpha \in S_N(V)$ . If  $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$  is the longest word then

$$\Delta_{\alpha_1} \circ \dots \circ \Delta_{\alpha_N} = d^{-1}((-1)^N w_0 + \sum_{w \neq w_0} q_w w)$$

where  $q_w \in \overline{S(V)}$ .

*Proof.* We compute

$$\begin{aligned} \Delta_{\alpha_1} \dots \Delta_{\alpha_N} &= \alpha_1^{-1} (1 - s_{\alpha_1}) \dots \alpha_N^{-1} (1 - s_{\alpha_N}) \\ &= (-1)^N \alpha_1^{-1} s_1 \dots \alpha_N^{-1} s_N + \sum_{w \neq w_0} a(w) w \end{aligned}$$

where the index of summation in the last term is a consequence of (1.4). It now suffices to watch in the first term what happens to inverted roots  $\alpha_i^{-1}$  as we pass the fundamental reflections  $s_i$  over to the right. Using (2.2 i) we get

$$(-1)^N \left( \prod_{i=1}^n s_{\alpha_1} \dots s_{\alpha_{i-1}} (\alpha_i) \right)^{-1} s_{\alpha_1} \dots s_{\alpha_N}$$

But by (1.3) this is  $(-1)^N d^{-1} w_0$  since  $w_0^{-1} = w_0$  converts all positive roots into negative roots by (1.4). We now let  $q_w = da_w$  and the proof is complete.

LEMMA 2.4. If  $f \in \text{End}(S(V))$  reduces the grading by  $N$  then  $d \cdot f = \lambda J$  for some  $\lambda \in \mathbf{R}$ , where  $J = \sum_{w \in W} (-1)^{l(w)} w$ .

*Proof.* See [11, Prop. 1 (b)].

PROPOSITION 2.5: If  $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$  as above, then  $\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = d^{-1} J$ .

*Proof.* By (2.4),  $d\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = \lambda J = \lambda(-1)^N w_0 + \sum_{w \neq w_0} (-1)^{l(w)} \lambda w$ .

Also by (2.3)  $d\Delta_{\alpha_1} \dots \Delta_{\alpha_N} = (-1)^N w_0 + \sum_{w \neq w_0} q_w w$ . By Dedekind's theorem (see, e.g., [1]) the  $w$ 's are independent as automorphisms of  $\overline{S(V)}$ , so  $\lambda = 1$  and the result follows.

We can now show

PROPOSITION 2.6.  $\Delta_w$  is well defined.

*Proof.* By [6, IV § 1, Prop. 5], it suffices to show

$$\Delta_\alpha \Delta_\beta \Delta_\alpha \dots = \Delta_\beta \Delta_\alpha \Delta_\beta \dots$$

with  $m_{\alpha\beta}$  terms on each side. But the dihedral root system  $I_2(m_{\alpha\beta})$  or  $A_1 \times A_1$  has  $s_\alpha s_\beta s_\alpha \dots = s_\beta s_\alpha s_\beta \dots$  as its longest word and hence (2.5) completes the argument.

THEOREM 2.7. The  $\{\Delta_w\}_{w \in W}$  are an  $S(V)$ -basis for  $\Delta_W$  and hence the  $\{\varepsilon \circ \Delta_w\}_{w \in W}$  are an  $\mathbf{R}$ -basis for  $\overline{\Delta}_W$ .

*Proof.* By (2.2 v), it is easy to check the  $\Delta_w$ 's generate  $\Delta_W$  as an  $S(V)$ -module. The linear independence follows from Dedekind's theorem, and the last statement is immediate.

We now define  $\{X_w\}_{w \in W}$  to be the basis of  $H_W = \overline{\Delta}_W^*$  dual to the basis  $\{\varepsilon \cdot \Delta_w\}_{w \in W}$  of  $\overline{\Delta}_W$ , i.e.

$$X_w(\varepsilon \cdot \Delta_{w'}) = \delta_{ww'}$$

This immediately yields the following "coordinate-wise" description of  $c$ .

$$c(u) = \sum_{w \in W} \varepsilon \Delta_w(u) X_w.$$

First, we show  $c$  has the correct kernel. We need the following Lemma that follows from R. Steinberg [21].

LEMMA 2.8. If  $I$  is a graded ideal of  $S(V)$  such that  $I_W \subseteq I$  and  $Rd \cap I = 0$ , then  $I = I_W$ .

This result rapidly yields our version of the "basis theorem" of the Schubert calculus, namely

**THEOREM 2.9.**  $\text{Ker}(c) = I_W$  and  $c$  induces an isomorphism  $S_W \approx H_W$ .

*Proof.* For the first assertion, by (2.8), it suffices to compute

$$\begin{aligned} c(\lambda d) &= \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0} \\ &= \lambda |W| X_{w_0}. \end{aligned}$$

Finally,  $c$  is clearly onto by construction.

In the next section we will work on producing an explicit section for  $c$ .

*Remark.* Demazure's proof, though restricted to Weyl groups, is done integrally. In that situation,  $c$  is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when  $W$  preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

### 3. GIAMBELLI FORMULA

We begin with an easy lemma.

**LEMMA 3.1.**  $\Delta$  is quasi-multiplicative, i.e.

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The first clause is immediate since the conditions implies that reduced decompositions of  $w$  and  $w'$  can be juxtaposed to yield a reduced decomposition of  $ww'$ . Now suppose  $w = s_\alpha w'$  and  $l(s_\alpha w') = l(w') - 1$  (that this is the only possibility that follows from (1.1)). Then  $w' = s_\alpha(s_\alpha w')$  and

$$l(w') = 1 + (l(w') - 1) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part  $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$ . But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on  $l(w)$  completes the proof.