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**Autor:** Hiller, Howard L.  
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## 1. COXETER GROUPS

We begin by reviewing some of the elementary theory of Coxeter groups. Some detail is included to avoid any oblique use of the crystallographic condition. Following Bourbaki [6, IV] we say  $(W, S)$  is a *finite Coxeter system* if  $W$  is a finite group given by the presentation  $\langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$  where  $m_{ij}$  is the order of  $s_i s_j$ . It is possible [6, V] to construct a real Euclidean space  $V$  and a root system  $(\Delta, \Sigma)$  in  $V$  that “geometrically realizes”  $(W, S)$ . By this we mean the following. If  $\gamma \in \Delta$  then

$$s_\gamma(x) = x - (x, \gamma^v) \gamma \quad \left( \text{co-root } \gamma^v = \frac{2\gamma}{(\gamma, \gamma)} \right)$$

is the reflection through the hyperplane perpendicular to the root  $\gamma$ , and we can form the subgroup  $W(\Delta)$  of  $GL(V)$  generated by the  $s_\gamma$ 's,  $\gamma \in \Delta$ . In fact, the  $s_\alpha$ 's,  $\alpha \in \Sigma$ , generate  $W(\Delta)$  and we call the pair  $(W(\Delta), \{s_\alpha : \alpha \in \Sigma\})$  the *Weyl system* of  $(\Delta, \Sigma)$ . Coxeter [9] proved that the Weyl system is always a Coxeter system and if this pair is isomorphic (in the obvious sense) to  $(W, S)$  we say  $(\Delta, \Sigma)$  is a *geometric realization* of  $(W, S)$ . Of course, the choice of such a  $(\Delta, \Sigma)$  is not unique. But clearly up to a rigid motion of  $V$ , the root system is determined by the lengths of the simple roots.

If the lengths can be chosen so that  $(\alpha, \beta^v) \in \mathbf{Z}$  for all  $\alpha, \beta \in \Sigma$ , we say  $W$  is *crystallographic* (or a Weyl group). Geometrically, this means that the  $\mathbf{Z}$ -lattice generated by  $\Sigma$  is preserved by  $W$ . As mentioned in the introduction, even this choice of relative lengths is not necessarily unique.



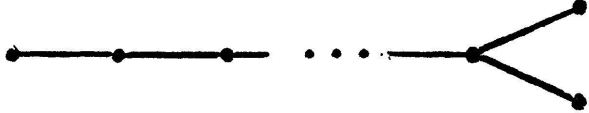
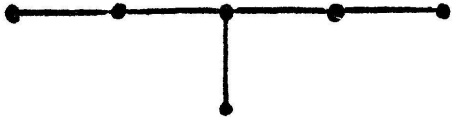
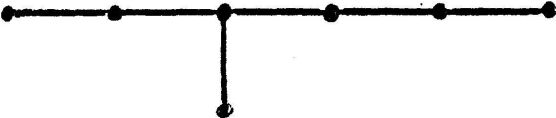

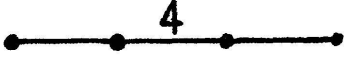
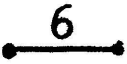


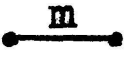
We can choose a vector  $t \in V$ , such that  $(t, \alpha) > 0$  for all  $\alpha \in \Sigma$  (i.e.  $t$  is in the fundamental chamber  $C$ ). This vector decomposes the roots  $\Delta = \Delta^+ \amalg \Delta^-$  where

$$\Delta^+ = \{\gamma \in \Delta : (\gamma, t) > 0\}$$

and  $\Delta^- = -\Delta^+$ . Note that  $|\Delta^+| = N = \frac{1}{2} |\Delta|$ , where  $N$  is the number of reflections in  $W$  as described in the introduction.

It is now customary to attach an edge labelled graph to  $(W, S)$  called the Coxeter graph. The nodes correspond to the elements of  $S$  and  $s_i$  is attached to  $s_j$  by an edge if  $m_{ij} \geq 3$ , and if also  $m_{ij} > 3$  the edge is labelled with the number  $m_{ij}$ . In 1934, Coxeter [9] classified the Coxeter groups with connected graphs and showed that every Coxeter group is a product of the “connected” components. The classification of the irreducible Coxeter groups along with the fundamental degrees is

TABLE

$W$	Coxeter graph	$d_1, \dots, d_n$
$A_n$		2, 3, ..., $n + 1$
$B_n$		2, 4, ..., $2n$
$D_n$		2, 4, ..., $2(n-2)$ , $2(n-1)$ , $n$
$E_6$		2, 5, 6, 8, 9, 12
$E_7$		2, 6, 8, 10, 12, 14, 18
$E_8$		2, 8, 12, 14, 18, 20, 24, 30
$F_4$		2, 6, 8, 12
$G_2$		2, 6
$H_3$		2, 6, 10
$H_4$		2, 12, 20, 30
$I_2(m)$		2, $m$

We will assume throughout that  $W$  is irreducible.

The crystallographic Coxeter groups and their root systems are well-known and correspond up to a choice of relative lengths of the simple roots

to the Cartan classification of simple Lie algebras over the complex numbers. The dihedral groups are the Weyl groups of  $I_2(m)$ , and are the symmetry groups of a regular  $m$ -gon (from which it is easy to construct  $(\Delta, \Sigma)$ ). The group  $H_3$  is isomorphic to a product of  $\mathbf{Z}_2$  and an alternating group on five letters and  $H_4$  is the symmetry group of a certain 4-dimensional polytope [9, 10].

The primary piece of structure available on a Coxeter group is the *length function*  $l: W \rightarrow \mathbf{N}$ , where  $l(w)$  is defined as the minimal length of an expression of  $w$  in the generators  $S$ . If  $l(w) = k$  and  $w = s_1 \dots s_k$ ,  $s_i \in S$ , we call this a *reduced decomposition* of  $W$ . There is an alternative intrinsic description.

LEMMA 1.1. *Let  $\Gamma_w$  denote the set of  $\gamma \in \Delta^+$  such that  $w(\gamma) \in \Delta^-$ , then*

- (i)  $|\Gamma_{ws_\alpha}| = |\Gamma_w| \pm 1$  if and only if  $w(\alpha) \in \Delta^\pm$ ,
- (ii)  $l(w) = |\Gamma_w|$ ,
- (iii)  $l(ws_\alpha) = l(w) \pm 1$  if and only if  $w(\alpha) \in \Delta^\pm$ .

*Proof.* To see (i) one need only recall that  $\Gamma_{s_\alpha} = \{\alpha\}$ . This first assertion then implies  $|\Gamma_w| \leq l(w)$ . The other inequality follows from an induction on  $|\Gamma_w|$  and then (ii) follows. (iii) is immediate from (i) and (ii).

The next piece of structure on the Coxeter group we require is the so-called Bruhat ordering [13]. We define  $w' \rightarrow w$  (intuitively,  $w'$  is an immediate predecessor of  $w$  if there exists a positive root  $\gamma$  such that  $\sigma_\gamma w = w'$  and  $l(w') = l(w) - 1$ . (We will often adorn  $\rightarrow$  with the unique such  $\gamma$ .) Since  $W$  is transitive on the roots and  $ws_\alpha w^{-1} = s_{w(\alpha)}$  the first condition is equivalent to  $w' w^{-1}$  being a conjugate of a fundamental reflection  $s \in S$ . The Bruhat order  $<$  on  $W$  is the transitive closure of the ordering  $\rightarrow$ . Note that  $l$  is forced to be strictly order-preserving so that the two pieces of structure we have introduced are compatible. We can now relate  $\rightarrow$  to any particular reduced decomposition of  $w$ .

LEMMA 1.2. *If  $w = s_1 \dots s_k$  is a reduced decomposition, then  $w' \rightarrow w$  only if  $w' = w_i^\wedge$  where  $w_i^\wedge = s_1 \dots \hat{s}_i \dots s_k$  (and  $\hat{\phantom{x}}$  denotes deletion).*

*Proof.* See Theorem 1.1 (III) in [13].

Hence, in general, the Bruhat ordering corresponds to the subwords of any reduced decomposition. So, for any  $i$  we can find a  $\gamma \in \Delta^+$  such that  $s_\gamma w_i^\wedge = w$ . The next result describes these roots  $\gamma$  both specifically and abstractly.

LEMMA 1.3. If  $w = s_1 \dots s_k$  is a reduced decomposition, define  $\theta_i = s_1 \dots s_{i-1}(\alpha_i)$  where  $s_i = s_{\alpha_i}$ ,  $\alpha_i \in \Sigma$ . Then the following sets are equal

- (i)  $\Gamma_{w^{-1}} = \Delta^+ \cap w(\Delta^-)$ ,
- (ii)  $\{\theta_i\}_{1 \leq i \leq k}$ ,
- (iii)  $\{\gamma \in \Delta^+ : s_\gamma w_i = w\}$ .

*Proof.* (i)  $\subseteq$  (ii). Let  $\gamma \in \Delta^+$  and  $w^{-1}(\gamma) \in \Delta^-$ . Let  $j$  be the smallest number such that  $s_j \dots s_1(\gamma) \in \Delta^-$ . Then  $\alpha_j = s_{j-1} \dots s_1(\gamma)$ . Hence  $\gamma = \theta_j$ .

(ii)  $\subseteq$  (iii). It suffices to compute

$$\begin{aligned} s_{\theta_i} \hat{w}_i &= s_{s_1} \dots s_{s_{i-1}}(\alpha_i)(s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_k = w. \end{aligned}$$

But now  $|\Gamma_{w^{-1}}| = l(w^{-1}) = l(w) = k$ , by (1.1) and certainly  $|\{\gamma \in \Delta^+ : s_\gamma \hat{w}_i = w\}| \leq k$ , so all three sets must be equal.

*Remark.* Though the  $\theta_i$ 's are defined in terms of a reduced decomposition, (1.3 i) shows that they are actually independent of the choice made.

We now recall that the Bruhat order on  $W$  possesses a unique top element of greatest length.

LEMMA 1.4. There exist a unique element  $w_0 \in W$  such that  $l(w_0) = N$ . In addition,  $w_0 \geq w$ , for all  $w \in W$ ,  $w_0^2 = 1$  and  $l(w w_0) = l(w_0) - l(w)$ .

*Proof.* One knows that  $W$  acts simply transitively on the chambers and  $w_0$  is chosen to be the unique element satisfying  $w_0 C = -C$ . The rest is standard, see [6, p. 43].

Finally, we make some remarks on the (anti) invariant theory of Coxeter groups. The main result is

PROPOSITION 1.5. If  $(W, S)$  is a Coxeter system, then the invariant algebra  $S(V)^W$  has  $|S|$  algebraically independent generators of degrees  $2 = d_1, d_2, \dots, d_n$ . Equivalently,  $S(V)$  is a free  $S(V)^W$ -module.

*Proof.* This follows immediately from Chevalley's theorem [8].

*Remark.* It is often useful in this context to think of  $W$  as the Galois group of the rational function field  $\overline{S(V)}$  over the rational function field  $\overline{S(V)^W}$  of the invariants. We exploit this point of view in the next section.

There is also a theory of anti-invariants, i.e. polynomials  $u \in S(V)$  such that  $w \cdot u = (-1)^{l(w)} u$ . The algebra of anti-invariants is written  $S(V)^{-W}$ . It is a free module of rank 1 over  $S(V)^W$  generated by the element  $d = \prod_{\gamma \in \Delta^+} \gamma \in S_N(V)$ . The corresponding “anti-averaging” operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

## 2. DEMAZURE'S BASIS THEOREM

Let  $\varepsilon : S(V) \rightarrow S_0(V) \approx \mathbf{R}$  denote the projection map. We begin by defining certain operators on  $S(V)$ , whose composition with  $\varepsilon$  should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let  $(W, S)$  be a Coxeter system and  $(\Delta, \Sigma)$  a geometric realization of it.

*Definition 2.1.* If  $\alpha \in \Delta$ , define  $\Delta_\alpha = \alpha^{-1} (1 - s_\alpha)$  as an  $S(V)^W$ -endomorphism of  $S(V)$ . (Note the division is legitimate since  $s_\alpha$  is the identity on the  $\ker(\alpha) = \alpha^\perp$ ; thinking of  $\alpha$  as a linear form  $x \mapsto (x, \alpha)$  in  $V^* = S_1(V)$ , of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

LEMMA 2.2. *If  $w \in W, \alpha \in \Delta, u, v \in S(V)$  then*

- (i)  $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$ ,
- (ii)  $\Delta_\alpha^2 = 0$ ,
- (iii)  $s_\alpha = 1 - \alpha \Delta_\alpha$ ,
- (iv)  $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$  (where the superscript denotes invariants)
- (v)  $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$ ,
- (vi)  $\Delta_\alpha(I_W) \subset I_W$ ,
- (vii)  $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^\vee, \omega) s_\alpha$ ,

where  $\omega^*$  denotes the operator multiplication by  $\omega$ .

We now define  $\Delta_W$  to be the subalgebra of the algebra of endomorphisms  $\text{End}(S(V))$  generated by the  $\Delta_\alpha$ 's ( $\alpha \in \Delta$ ) and  $\omega^*, \omega \in S(V)$ . Note  $\Delta_\alpha$  decreases the grading by  $(-1)$  and  $W \subseteq \Delta_W$  by (2.2 iii).