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1. COXETER GROUPS

We begin by reviewing some of the elementary theory of Coxeter groups. Some detail is included to avoid any oblique use of the crystallographic condition. Following Bourbaki [6, IV] we say (W, S) is a *finite Coxeter system* if W is a finite group given by the presentation $\langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where m_{ij} is the order of $s_i s_j$. It is possible [6, V] to construct a real Euclidean space V and a root system (Δ, Σ) in V that “geometrically realizes” (W, S) . By this we mean the following. If $\gamma \in \Delta$ then

$$s_\gamma(x) = x - (x, \gamma^v) \gamma \quad \left(\text{co-root } \gamma^v = \frac{2\gamma}{(\gamma, \gamma)} \right)$$

is the reflection through the hyperplane perpendicular to the root γ , and we can form the subgroup $W(\Delta)$ of $GL(V)$ generated by the s_γ 's, $\gamma \in \Delta$. In fact, the s_α 's, $\alpha \in \Sigma$, generate $W(\Delta)$ and we call the pair $(W(\Delta), \{s_\alpha : \alpha \in \Sigma\})$ the *Weyl system* of (Δ, Σ) . Coxeter [9] proved that the Weyl system is always a Coxeter system and if this pair is isomorphic (in the obvious sense) to (W, S) we say (Δ, Σ) is a *geometric realization* of (W, S) . Of course, the choice of such a (Δ, Σ) is not unique. But clearly up to a rigid motion of V , the root system is determined by the lengths of the simple roots.

If the lengths can be chosen so that $(\alpha, \beta^v) \in \mathbf{Z}$ for all $\alpha, \beta \in \Sigma$, we say W is *crystallographic* (or a Weyl group). Geometrically, this means that the \mathbf{Z} -lattice generated by Σ is preserved by W . As mentioned in the introduction, even this choice of relative lengths is not necessarily unique.



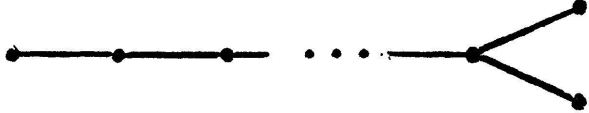
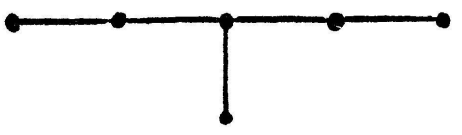
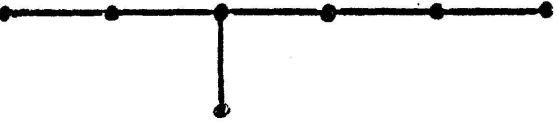
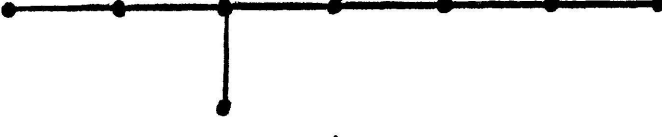
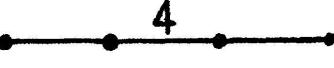
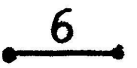



We can choose a vector $t \in V$, such that $(t, \alpha) > 0$ for all $\alpha \in \Sigma$ (i.e. t is in the fundamental chamber C). This vector decomposes the roots $\Delta = \Delta^+ \amalg \Delta^-$ where

$$\Delta^+ = \{\gamma \in \Delta : (\gamma, t) > 0\}$$

and $\Delta^- = -\Delta^+$. Note that $|\Delta^+| = N = \frac{1}{2} |\Delta|$, where N is the number of reflections in W as described in the introduction.

It is now customary to attach an edge labelled graph to (W, S) called the Coxeter graph. The nodes correspond to the elements of S and s_i is attached to s_j by an edge if $m_{ij} \geq 3$, and if also $m_{ij} > 3$ the edge is labelled with the number m_{ij} . In 1934, Coxeter [9] classified the Coxeter groups with connected graphs and showed that every Coxeter group is a product of the “connected” components. The classification of the irreducible Coxeter groups along with the fundamental degrees is

TABLE

W	Coxeter graph	d_1, \dots, d_n
A_n		$2, 3, \dots, n + 1$
B_n		$2, 4, \dots, 2n$
D_n		$2, 4, \dots, 2(n-2), 2(n-1), n$
E_6		$2, 5, 6, 8, 9, 12$
E_7		$2, 6, 8, 10, 12, 14, 18$
E_8		$2, 8, 12, 14, 18, 20, 24, 30$
F_4		$2, 6, 8, 12$
G_2		$2, 6$
H_3		$2, 6, 10$
H_4		$2, 12, 20, 30$
$I_2(m)$		$2, m$

We will assume throughout that W is irreducible.

The crystallographic Coxeter groups and their root systems are well-known and correspond up to a choice of relative lengths of the simple roots

to the Cartan classification of simple Lie algebras over the complex numbers. The dihedral groups are the Weyl groups of $I_2(m)$, and are the symmetry groups of a regular m -gon (from which it is easy to construct (Δ, Σ)). The group H_3 is isomorphic to a product of \mathbf{Z}_2 and an alternating group on five letters and H_4 is the symmetry group of a certain 4-dimensional polytope [9, 10].

The primary piece of structure available on a Coxeter group is the *length function* $l: W \rightarrow \mathbf{N}$, where $l(w)$ is defined as the minimal length of an expression of w in the generators S . If $l(w) = k$ and $w = s_1 \dots s_k$, $s_i \in S$, we call this a *reduced decomposition* of W . There is an alternative intrinsic description.

LEMMA 1.1. *Let Γ_w denote the set of $\gamma \in \Delta^+$ such that $w(\gamma) \in \Delta^-$, then*

- (i) $|\Gamma_{ws_\alpha}| = |\Gamma_w| \pm 1$ if and only if $w(\alpha) \in \Delta^\pm$,
- (ii) $l(w) = |\Gamma_w|$,
- (iii) $l(ws_\alpha) = l(w) \pm 1$ if and only if $w(\alpha) \in \Delta^\pm$.

Proof. To see (i) one need only recall that $\Gamma_{s_\alpha} = \{\alpha\}$. This first assertion then implies $|\Gamma_w| \leq l(w)$. The other inequality follows from an induction on $|\Gamma_w|$ and then (ii) follows. (iii) is immediate from (i) and (ii).

The next piece of structure on the Coxeter group we require is the so-called Bruhat ordering [13]. We define $w' \rightarrow w$ (intuitively, w' is an immediate predecessor of w if there exists a positive root γ such that $\sigma_\gamma w = w'$ and $l(w') = l(w) - 1$. (We will often adorn \rightarrow with the unique such γ .) Since W is transitive on the roots and $ws_\alpha w^{-1} = s_{w(\alpha)}$ the first condition is equivalent to $w' w^{-1}$ being a conjugate of a fundamental reflection $s \in S$. The Bruhat order $<$ on W is the transitive closure of the ordering \rightarrow . Note that l is forced to be strictly order-preserving so that the two pieces of structure we have introduced are compatible. We can now relate \rightarrow to any particular reduced decomposition of w .

LEMMA 1.2. *If $w = s_1 \dots s_k$ is a reduced decomposition, then $w' \rightarrow w$ only if $w' = w_i^\wedge$ where $w_i^\wedge = s_1 \dots \hat{s}_i \dots s_k$ (and $\hat{}$ denotes deletion).*

Proof. See Theorem 1.1 (III) in [13].

Hence, in general, the Bruhat ordering corresponds to the subwords of any reduced decomposition. So, for any i we can find a $\gamma \in \Delta^+$ such that $s_\gamma w_i^\wedge = w$. The next result describes these roots γ both specifically and abstractly.

LEMMA 1.3. If $w = s_1 \dots s_k$ is a reduced decomposition, define $\theta_i = s_1 \dots s_{i-1} (\alpha_i)$ where $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$. Then the following sets are equal

- (i) $\Gamma_{w^{-1}} = \Delta^+ \cap w(\Delta^-)$,
- (ii) $\{\theta_i\}_{1 \leq i \leq k}$,
- (iii) $\{\gamma \in \Delta^+ : s_\gamma w_i = w\}$.

Proof. (i) \subseteq (ii). Let $\gamma \in \Delta^+$ and $w^{-1}(\gamma) \in \Delta^-$. Let j be the smallest number such that $s_j \dots s_1(\gamma) \in \Delta^-$. Then $\alpha_j = s_{j-1} \dots s_1(\gamma)$. Hence $\gamma = \theta_j$.

(ii) \subseteq (iii). It suffices to compute

$$\begin{aligned} s_{\theta_i} \hat{w}_i &= s_{s_1} \dots s_{s_{i-1}} (\alpha_i) (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_k = w. \end{aligned}$$

But now $|\Gamma_{w^{-1}}| = l(w^{-1}) = l(w) = k$, by (1.1) and certainly $|\{\gamma \in \Delta^+ : s_\gamma \hat{w}_i = w\}| \leq k$, so all three sets must be equal.

Remark. Though the θ_i 's are defined in terms of a reduced decomposition, (1.3 i) shows that they are actually independent of the choice made.

We now recall that the Bruhat order on W possesses a unique top element of greatest length.

LEMMA 1.4. There exist a unique element $w_0 \in W$ such that $l(w_0) = N$. In addition, $w_0 \geq w$, for all $w \in W$, $w_0^2 = 1$ and $l(w w_0) = l(w_0) - l(w)$.

Proof. One knows that W acts simply transitively on the chambers and w_0 is chosen to be the unique element satisfying $w_0 C = -C$. The rest is standard, see [6, p. 43].

Finally, we make some remarks on the (anti) invariant theory of Coxeter groups. The main result is

PROPOSITION 1.5. If (W, S) is a Coxeter system, then the invariant algebra $S(V)^W$ has $|S|$ algebraically independent generators of degrees $2 = d_1, d_2, \dots, d_n$. Equivalently, $S(V)$ is a free $S(V)^W$ -module.

Proof. This follows immediately from Chevalley's theorem [8].

Remark. It is often useful in this context to think of W as the Galois group of the rational function field $\overline{S(V)}$ over the rational function field $\overline{S(V)^W}$ of the invariants. We exploit this point of view in the next section.

There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta^+} \gamma \in S_N(V)$. The corresponding “anti-averaging” operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

2. DEMAZURE'S BASIS THEOREM

Let $\varepsilon : S(V) \rightarrow S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1} (1 - s_\alpha)$ as an $S(V)^W$ -endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

LEMMA 2.2. *If $w \in W, \alpha \in \Delta, u, v \in S(V)$ then*

- (i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$,
- (ii) $\Delta_\alpha^2 = 0$,
- (iii) $s_\alpha = 1 - \alpha \Delta_\alpha$,
- (iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ (where the superscript denotes invariants)
- (v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$,
- (vi) $\Delta_\alpha(I_W) \subset I_W$,
- (vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^\nu, \omega) s_\alpha$,

where ω^* denotes the operator multiplication by ω .

We now define Δ_W to be the subalgebra of the algebra of endomorphisms $\text{End}(S(V))$ generated by the Δ_α 's ($\alpha \in \Delta$) and $\omega^*, \omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \Delta_W$ by (2.2 iii).