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1. Coxeter groups

We begin by reviewing some of the elementary theory of Coxeter groups. Some detail is included to avoid any oblique use of the crystallographic condition. Following Bourbaki [6, IV] we say (W, S) is a *finite Coxeter system* if W is a finite group given by the presentation $\langle s_i \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ where m_{ij} is the order of $s_i s_j$. It is possible [6, V] to construct a real Euclidean space V and a root system (Δ, Σ) in V that "geometrically realizes" (W, S). By this we mean the following. If $\gamma \in \Delta$ then

$$s_\gamma(x) = x - (x, \gamma^v)\gamma \qquad \left(\text{co-root} \quad \gamma^v = \frac{2\gamma}{(\gamma, \gamma)}\right)$$

is the reflection through the hyperplane perpendicular to the root γ, and we can form the subgroup $W(\Delta)$ of $GL(V)$ generated by the s_γ's, $\gamma \in \Delta$. In fact, the s_α's, $\alpha \in \Sigma$, generate $W(\Delta)$ and we call the pair $(W(\Delta), \{s_\alpha : \alpha \in \Sigma\})$ the *Weyl system* of (Δ, Σ). Coxeter [9] proved that the Weyl system is always a Coxeter system and if this pair is isomorphic (in the obvious sense) to (W, S) we say (Δ, Σ) is a geometric realization of (W, S). Of course, the choice of such a (Δ, Σ) is not unique. But clearly up to a rigid motion of V, the root system is determined by the lengths of the simple roots.

If the lengths can be chosen so that $(\alpha, \beta^v) \in \mathbf{Z}$ for all $\alpha, \beta \in \Sigma$, we say W is *crystallographic* (or a Weyl group). Geometrically, this means that the $\mathbf{Z}$-lattice generated by Σ is preserved by W. As mentioned in the introduction, even this choice of relative lengths is not necessarily unique.

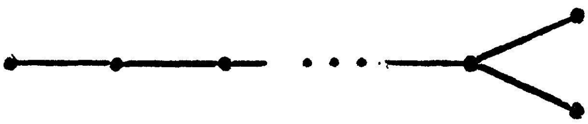
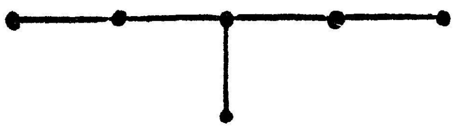
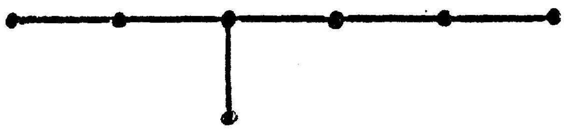
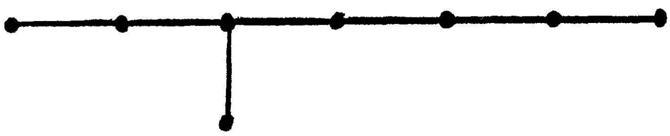
We can choose a vector $t \in V$, such that $(t, \alpha) > 0$ for all $\alpha \in \Sigma$ (i.e. t is in the fundamental chamber C). This vector decomposes the roots $\Delta = \Delta^+ \coprod \Delta^-$ where

$$\Delta^+ = \{\gamma \in \Delta : (\gamma, t) > 0\}$$

and $\Delta^- = -\Delta^+$. Note that $|\Delta^+| = N = \frac{1}{2}|\Delta|$, where N is the number of reflections in W as described in the introduction.

It is now customary to attach an edge labelled graph to (W, S) called the Coxeter graph. The nodes correspond to the elements of S and s_i is attached to s_j by an edge if $m_{ij} \geqslant 3$, and if also $m_{ij} > 3$ the edge is labelled with the number m_{ij}. In 1934, Coxeter [9] classified the Coxeter groups with connected graphs and showed that every Coxeter group is a product of the "connected" components. The classification of the irreducible Coxeter groups along with the fundamental degrees is

TABLE

W	Coxeter graph	$d_1, \ldots, d_n$
A_n		2, 3, ..., $n+1$
B_n		2, 4, ..., $2n$
D_n		2, 4, ..., $2(n-2)$, $2(n-1)$, n
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6
H_3		2, 6, 10
H_4		2, 12, 20, 30
$I_2(m)$		2, m

We will assume throughout that W is irreducible.

The crystallographic Coxeter groups and their root systems are well-known and correspond up to a choice of relative lengths of the simple roots to the Cartan classification of simple Lie algebras over the complex numbers. The dihedral groups are the Weyl groups of $I_2(m)$, and are the symmetry groups of a regular m-gon (from which it is easy to construct (Δ, Σ)). The group H_3 is isomorphic to a product of $\mathbf{Z}_2$ and an alternating group on five letters and H_4 is the symmetry group of a certain 4-dimensional polytope [9, 10].

The primary piece of structure available on a Coxeter group is the *length function* $l : W \to \mathbf{N}$, where $l(w)$ is defined as the minimal length of an expression of w in the generators S. If $l(w) = k$ and $w = s_1 \dots s_k$, $s_i \in S$, we call this a *reduced decomposition* of W. There is an alternative intrinsic description.

LEMMA 1.1. *Let* Γ_w *denote the set of* $\gamma \in \Delta^+$ *such that* $w(\gamma) \in \Delta^-$, *then*

(i) $|\Gamma_{ws_\alpha}| = |\Gamma_w| \pm 1$ *if and only if* $w(\alpha) \in \Delta^\pm$,

(ii) $l(w) = |\Gamma_w|$,

(iii) $l(ws_\alpha) = l(w) \pm 1$ *if and only if* $w(\alpha) \in \Delta^\pm$.

Proof. To see (i) one need only recall that $\Gamma_{s_\alpha} = \{\alpha\}$. This first assertion then implies $|\Gamma_w| \leqslant l(w)$. The other inequality follows from an induction on $|\Gamma_w|$ and then (ii) follows. (iii) is immediate from (i) and (ii).

The next piece of structure on the Coxeter group we require is the so-called Bruhat ordering [13]. We define $w' \to w$ (intuitively, w' is an immediate predecessor of w if there exists a positive root γ such that $\sigma_\gamma w = w'$ and $l(w') = l(w) + 1$. (We will often adorn $\to$ with the unique such γ.) Since W is transitive on the roots and $ws_\alpha w^{-1} = s_{w(\alpha)}$ the first condition is equivalent to $w' w^{-1}$ being a conjugate of a fundamental reflection $s \in S$. The *Bruhat order* $<$ on W is the transitive closure of the ordering $\to$. Note that l is forced to be strictly order-preserving so that the two pieces of structure we have introduced are compatible. We can now relate $\to$ to any particular reduced decomposition of w.

LEMMA 1.2. *If* $w = s_1 \dots s_k$ *is a reduced decomposition, then* $w' \to w$ *only if* $w' = w_{\hat{i}}$ *where* $w_{\hat{i}} = s_1 \dots \hat{s}_i \dots s_k$ *(and* $\hat{}$ *denotes deletion).*

Proof. See Theorem 1.1 (III) in [13].

Hence, in general, the Bruhat ordering corresponds to the subwords of any reduced decomposition. So, for any i we can find a $\gamma \in \Delta^+$ such that $s_\gamma w_{\hat{i}} = w$. The next result describes these roots γ both specifically and abstractly.

Lemma 1.3. *If* $w = s_1 \dots s_k$ *is a reduced decomposition, define* $\theta_i = s_1 \dots s_{i-1}(\alpha_i)$ *where* $s_i = s_{\alpha_i}, \alpha_i \in \Sigma$. *Then the following sets are equal*

(i) $\Gamma_{w^{-1}} = \Delta^+ \cap w(\Delta^-)$,

(ii) $\{\theta_i\}_{1 \leq i \leq k}$,

(iii) $\{\gamma \in \Delta^+ : s_\gamma w_{\hat{i}} = w\}$.

Proof. (i) $\subseteq$ (ii). Let $\gamma \in \Delta^+$ and $w^{-1}(\gamma) \in \Delta^-$. Let j be the smallest number such that $s_j \dots s_1(\gamma) \in \Delta^-$. Then $\alpha_j = s_{j-1} \dots s_1(\gamma)$. Hence $\gamma = \theta_j$.

(ii) $\subseteq$ (iii). It suffices to compute

$$\begin{aligned} s_{\theta_i} w_{\hat{i}} &= s_{s_1 \dots s_{i-1}(\alpha_i)} (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 (s_1 \dots \hat{s}_i \dots s_k) \\ &= s_1 \dots s_k = w\,. \end{aligned}$$

But now $|\Gamma_{w^{-1}}| = l(w^{-1}) = l(w) = k$, by (1.1) and certainly $|\{\gamma \in \Delta^+ : s_\gamma w_{\hat{i}} = w\}| \leq k$, so all three sets must be equal.

Remark. Though the θ_i's are defined in terms of a reduced decomposition, (1.3 i) shows that they are actually independent of the choice made.

We now recall that the Bruhat order on W possesses a unique top element of greatest length.

Lemma 1.4. *There exist a unique element* $w_0 \in W$ *such that* $l(w_0) = N$. *In addition,* $w_0 \geq w$, *for all* $w \in W$, $w_0^2 = 1$ *and* $l(ww_0) = l(w_0) - l(w)$.

Proof. One knows that W acts simply transitively on the chambers and w_0 is chosen to be the unique element satisfying $w_0 C = -C$. The rest is standard, see [6, p. 43].

Finally, we make some remarks on the (anti) invariant theory of Coxeter groups. The main result is

Proposition 1.5. *If* (W, S) *is a Coxeter system, then the invariant algebra* $S(V)^W$ *has* $|S|$ *algebraically independent generators of degrees* $2 = d_1, d_2, \dots, d_n$. *Equivalently,* $S(V)$ *is a free* $S(V)^W$*-module.*

Proof. This follows immediately from Chevalley's theorem [8].

Remark. It is often useful in this context to think of W as the Galois group of the rational function field $\overline{S(V)}$ over the rational function field $\overline{S(V)^W}$ of the invariants. We exploit this point of view in the next section.

There is also a theory of anti-invariants, i.e. polynomials $u \in S(V)$ such that $w \cdot u = (-1)^{l(w)} u$. The algebra of anti-invariants is written $S(V)^{-W}$. It is a free module of rank 1 over $S(V)^W$ generated by the element $d = \prod_{\gamma \in \Delta +} \gamma \in S_N(V)$. The corresponding "anti-averaging" operating is

$$\frac{1}{|W|} J(u) = \frac{1}{|W|} \sum_{w \in W} (-1)^{l(w)} w \cdot u.$$

2. Demazure's basis theorem

Let $\varepsilon : S(V) \to S_0(V) \approx \mathbf{R}$ denote the projection map. We begin by defining certain operators on $S(V)$, whose composition with ε should be thought of as algebraic models for Bruhat cells. To do this one must view the homology as a real functional on the cohomology via the usual pairing. The operators also admit an analytic interpretation [21]. As above, let (W, S) be a Coxeter system and (Δ, Σ) a geometric realization of it.

Definition 2.1. If $\alpha \in \Delta$, define $\Delta_\alpha = \alpha^{-1}(1-s_\alpha)$ as an $S(V)^W$-endomorphism of $S(V)$. (Note the division is legitimate since s_α is the identity on the $\ker(\alpha) = \alpha^\perp$; thinking of α as a linear form $x \mapsto (x, \alpha)$ in $V^* = S_1(V)$, of course.)

The following result summarizes the relevant properties of these operators and the proof is routine

Lemma 2.2. *If* $w \in W, \alpha \in \Delta, u, v \in S(V)$ *then*

(i) $w \Delta_\alpha w^{-1} = \Delta_{w(\alpha)}$,

(ii) $\Delta_\alpha^2 = 0$,

(iii) $s_\alpha = 1 - \alpha \Delta_\alpha$,

(iv) $\ker(\Delta_\alpha) = S(V)^{(s_\alpha)}$ *(where the superscript denotes invariants)*

(v) $\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v)$,

(vi) $\Delta_\alpha(I_W) \subset I_W$,

(vii) $[\Delta_\alpha, \omega^*] = \Delta_\alpha \omega^* - \omega^* \Delta_\alpha = (\alpha^v, \omega) s_\alpha$,

where ω^* *denotes the operator multiplication by* ω.

We now define $\mathbb{A}_W$ to be the subalgebra of the algebra of endomorphisms $\mathrm{End}(S(V))$ generated by the Δ_α's ($\alpha \in \Delta$) and ω^*, $\omega \in S(V)$. Note Δ_α decreases the grading by (-1) and $W \subseteq \mathbb{A}_W$ by (2.2 iii).