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$$\left| \sum_{0 \leq z < p^{n+1}} e(az^2 p^{-1} + hzp^{-n-1}) \right| = \begin{cases} 0 & p^n \nmid h \\ p^{n+\frac{1}{2}} & p^n \mid h, \quad p \nmid a \\ p^{n+1} & p^{n+1} \mid h, \quad p \mid a \\ 0 & p^{n+1} \nmid h, \quad p \mid a. \end{cases}$$

Proof: The first two parts of this lemma are Lemma 5 of [2]. The last two parts are trivial.

3. PROOF OF THEOREMS 1 AND 2

PROPOSITION 1. Let p be a prime, m a positive integer and a_1, \dots, a_k , integers such that

$$(a_1, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = p^h \quad 0 \leq h < m.$$

Then

$$S(a_1, \dots, a_k; p^m) = (p^h)^{k-1} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h})$$

Proof: The proof is similar to that of [2], page 85 bottom.

PROPOSITION 2. Let m, n be positive integers $\frac{1}{2}m \leq n < m$, p a prime, and a_1, \dots, a_k integers such that $(a_1, a_k; p^m) = 1$. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq A(p^n)^{k-1}$$

where

$$A = \begin{cases} k & \text{if } p > 2. \\ 1 & \text{if } p = 2 \text{ and } k \text{ is odd.} \\ \min \{ 2^{r+1}, p^m \} & \text{if } p = 2 \text{ and } k = 2^r l, \\ & r > 1 \text{ and } l \text{ odd.} \end{cases}$$

Proof: Let us assume throughout this proposition that $S(a_1, \dots, a_k; p^m) \neq 0$, or else we are done.

Now we have the identity

$$\sum_{\substack{0 < x_1, \dots, x_{k-1} \leq p^m \\ p \nmid x_1, \dots, p \nmid x_{k-1}}} f(x_1, \dots, x_{k-1})$$

$$= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^{m-n}} f(y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}).$$

Letting

$$f(x_1, \dots, x_{k-1}) = e\left(\frac{a_1 x_1 + \dots + a_{k-1} x_{k-1} + a_k [x_1, \dots, x_{k-1}; p^m]}{p^m}\right)$$

we see, using Lemma 2

$$= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} S(a_1, \dots, a_k; p^m) e\left(\frac{a_1 y_1 + \dots + a_{k-1} y_{k-1} + a_k [y_1, \dots, y_{k-1}; p^m]}{p^m}\right)$$

$$\sum_{0 \leq z_1 < p^{m-n}} e(\{a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m]\} p^{n-m} z_1)$$

$$\vdots$$

$$\sum_{0 \leq z_{k-1} < p^{m-n}} e(\{a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2\} p^{n-m} z_{k-1}).$$

Now since we have assumed $S(a_1, \dots, a_k; p^m) \neq 0$, the inner sums above must not equal 0. Thus

$$p^{m-n} \mid a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m]$$

$$\vdots$$

$$p^{m-n} \mid a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2.$$

These congruences imply, since $(a_1, a_k, p^m) = 1$, also $(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = 1$, and moreover

$$p \nmid a_1, \dots, p \nmid a_k.$$

Now we have

$$\begin{aligned}
& \leq (p^{m-n})^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} |S(a_1, \dots, a_k; p^m)| \\
& \quad a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{m-n}} \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^{m-n}} \\
& = (p^n)^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-n} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} |S(a_1, \dots, a_k; p^m)| \\
& \quad a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{m-n}} \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^{m-n}}
\end{aligned}$$

Now the congruences in the above sum are easily seen to be equivalent to :

$$\begin{aligned}
a_1 y_1 &\equiv a_2 y_2 \equiv \dots \equiv a_{k-1} y_{k-1} \pmod{p^{m-n}} \\
y_1^k &\equiv [a_1; p^m]^{k-1} a_2 \dots a_k \pmod{p^{m-n}}.
\end{aligned}$$

Thus by Lemma 1, the proposition is proved.

PROPOSITION 3. *Let $p > 2$ be a prime, a_1, \dots, a_k integers such that $(a_1, a_k, p^m) = 1$, where m is a positive even integer. Then*

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}}$$

Proof: This is Proposition 2, with $n = \frac{m}{2}$.

PROPOSITION 4. Let $p = 2$, m a positive integer, a_1, \dots, a_k integers such that $(a_1, a_k, p^m) = 1$. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq A_1 (p^m)^{\frac{k-1}{2}}$$

where

$$A_1 = \begin{cases} 1 & , \text{ if } m \text{ even, } k \text{ odd.} \\ \min \{ 2^{r+1}, p^m \} & , \text{ if } m \text{ even, } k = 2^r l, r > 1, l \text{ odd.} \\ 2^{\frac{k-1}{2}} & , \text{ if } m \text{ odd, } k \text{ odd.} \\ 2^{\frac{k-1}{2}} \min \{ 2^{r+1}, p^m \} & , \text{ if } m \text{ odd, } k = 2^r l, r > 1, l \text{ odd.} \end{cases}$$

Proof: This follows from Proposition 2 with $n = m - [\frac{1}{2}m]$.

PROPOSITION 5. Let $p > 2$ be a prime, a_1, \dots, a_k integers. Then

$$|S(a_1, \dots, a_k; p)| \leq k p^{\frac{k-1}{2}} (a_1, a_k; p)^{1/2} \dots (a_{k-1}, a_k, p)^{1/2}$$

Proof: If $p \nmid a_1 \dots a_k$ this is Deligne's theorem. Therefore suppose, without loss of generality that $p \mid a_k, \dots, p \mid a_{k-i+1}$ where $i \geq 1$. Thus:

$$\begin{aligned} S(a_1, \dots, a_k; p) &= (p-1)^{i-1} \sum_{0 < x_1 < p} e\left(\frac{a_1 x_1}{p}\right) \dots \sum_{0 < x_{k-i} < p} e\left(\frac{a_{k-i} x_{k-i}}{p}\right) \\ &= (p-1)^{i-1} (-1)^{k-i} \end{aligned}$$

and so the proposition is proved.

PROPOSITION 6. Let $p > 2$ be a prime and $m > 1$ an odd positive integer. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

Proof: Let us assume throughout this proposition that $S(a_1, \dots, a_k; p^m) \neq 0$, or else we are done. Let $\frac{m-1}{2} = n > 0$.

Now we have the identity:

$$\begin{aligned}
& \sum_{\substack{0 < x_1, \dots, x_{k-1} \leq p^{2n+1} \\ p \nmid x_1, \dots, p \nmid x_{k-1}}} f(x_1, \dots, x_{k-1}) \\
= & \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^{n+1}} f(y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}).
\end{aligned}$$

Letting

$$f(x_1, \dots, x_{k-1}) = e\left(\frac{a_1 x_1 + \dots + a_{k-1} x_{k-1} + a_k [x_1, \dots, x_{k-1}; p^m]}{p^m}\right)$$

we see, using Lemma 3

$$\begin{aligned}
& S(a_1, \dots, a_k; p^m) \\
= & \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} e\left(\frac{a_1 y_1 + \dots + a_{k-1} y_{k-1} + a_k [y_1, \dots, y_{k-1}; p^m]}{p^m}\right) \\
& \sum_{0 \leq z_{k-1} < p^{n+1}} e(\{a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2\} p^{-n-1} z_{k-1} \\
& \quad + [y_1; p^m] \dots [y_{k-1}; p^m]^3 a_k p^{-1} z_{k-1}^2) \\
& \cdot \\
& \cdot \\
& \cdot \\
& \sum_{0 \leq z_1 < p^{n+1}} e(\{a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \\
& \quad + a_k [y_1; p^m]^2 [y_2; p^m]^2 \dots [y_{k-1}; p^m] z_2 p^n \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad + a_k [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m]^2 z_{k-1} p^n\} p^{-n-1} z_1 \\
& \quad + a_k [y_1; p^m]^3 \dots [y_{k-1}; p^m] z_2^1 p^{-1})
\end{aligned}$$

Since $S(a_1, \dots, a_k, p^m)$ is assumed to be non-zero, we see by Lemma 4 that:

$$\begin{aligned}
 p^n \mid & \{ a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \\
 & + a_k [y_1; p^m]^2 [y_2; p^m]^2 \dots [y_{k-1}; p^m] z_2 p^n \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + a_k [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m]^2 z_{k-1} p^n \} \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 p^n \mid & \{ a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \} .
 \end{aligned}$$

Now let us assume $(a_1, a_k, p^m) = 1$. By reasoning similar to that of Proposition 2, we see that

$$(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = 1 ,$$

and that $p \nmid a_k$. Thus by Lemma 4:

$$\begin{aligned}
 & | S(a_1, \dots, a_k; p^m) | \\
 \leq & (p^{n+1/2})^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} 1 \\
 a_1 & \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^n} \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 a_{k-1} & \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^n}
 \end{aligned}$$

Now by reasoning as in Proposition 2 we see $p \nmid a_1, \dots, p \nmid a_{k-1}$, and so by Lemma 1:

$$| S(a_1, \dots, a_k; p^m) | \leq k p^{(n+1/2)k-1} .$$

Now let us assume

$$(a_1, a_k, p^m) = p^h, \quad 0 < h < n + 1 ,$$

(if this case is possible.)

Thus $p \mid a_k$, and Lemma 4 now shows:

$$a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{n+1}}$$

\cdot
\cdot
\cdot

$$a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m] \pmod{p^{n+1}}.$$

Thus:

$$(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = p^h.$$

Thus by Proposition 1, we have:

$$S(a_1, \dots, a_k; p^m) = p^{(k-1)h} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h}).$$

Now by Proposition 3, 5 and the first part of this proposition, we have:

$$|S(a_1, \dots, a_k; p^m)| \leq k p^{(k-1)h} (p^{m-h})^{\frac{k-1}{2}} = k (p^m)^{\frac{k-1}{2}} (p^h)^{\frac{k-1}{2}}.$$

Now let us assume

$$(a_1, a_k, p^m) = p^{h_1}, \quad h_1 \geq n+1.$$

As in the previous argument we see

$$(a_2, a_k, p^m) = p^{h_2}, \quad h_2 \geq n+1$$

\cdot
\cdot
\cdot

$$(a_{k-1}, a_k, p^m) = p^{h_{k-1}}, \quad h_{k-1} \geq n+1.$$

Let $h = \min \{h_1, \dots, h_{k-1}\}$. We may assume $h < m$ or else the result is trivial. Now

$$\begin{aligned} & S(a_1, \dots, a_k; p^m) \\ &= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-h} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^h} e \\ & \left(\frac{a_1 (y_1 + p^{m-h} z_1) + \dots + a_k [y_1 + p^{m-h} z_1, \dots; p^m]}{p^m} \right). \end{aligned}$$

Now since $p^m \mid a_1 p^{m-h}, \dots, p^m \mid a_{k-1} p^{m-h}$ and since

$$[y_1 + p^{m-h} z_1, \dots, y_{k-1} + p^{m-h} z_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^{m-h}] \pmod{p^{m-h}}$$

we have

$$\begin{aligned}
 & S(a_1, \dots, a_k; p^m) \\
 = & p^{h(k-1)} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-h} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} e^{\left(\frac{a_1 p^{-h} y_1 + \dots + a_k p^{-h} [y_1, \dots, y_{k-1}; p^{m-h}]}{p^{m-h}} \right)} \\
 & = p^{h(k-1)} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h}).
 \end{aligned}$$

Now we may assume without loss of generality that $h = h_1$. Thus by Propositions 3, 5 and the first part of this proposition,

$$\begin{aligned}
 |S(a_1, \dots, a_k; p^m)| & \leq k p^{h(k-1)} p^{(m-h)\binom{k-1}{2}} \\
 & = k (p^m)^{\frac{k-1}{2}} (p^h)^{\frac{k-1}{2}}.
 \end{aligned}$$

PROPOSITION 7. Let $p > 2$ be a prime, m an even positive integer, and a_1, \dots, a_k integers. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

Proof: Using the identity of Proposition 2 and the results of Propositions 3, 5, 6, this is proved as Proposition 6.

PROPOSITION 8. Let $p = 2$, m a positive integer, a_1, \dots, a_k integers. Then

$$|S(a_1, \dots, a_k; p^m)| \leq 2^{\frac{k+1}{2}} k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

Proof: This is proved as Proposition 7.

THEOREM 1. Let q be a positive odd integer. Then for any integers a_1, \dots, a_k :

$$|S(a_1, \dots, a_k; q)| \leq k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}.$$

Proof: We proceed by induction on q . For $q = 1$ the theorem is trivial. Assume the theorem true for all $S(b_1, \dots, b_k; q')$, $q' < q$, b_1, \dots, b_k integers.

Now consider $S(a_1, \dots, a_k; q)$.

By Propositions 5, 6, 7, we may assume q is not a prime power; hence there exist odd q_1, q_2 such that $q = q_1 q_2$, $(q_1, q_2) = 1$, $q_1 > 1$, $q_2 > 1$. Thus there exist integers a_{k_1}, a_{k_2} such that

$$a_k = a_{k_1} q_2^k + a_{k_2} q_1^k.$$

By the multiplicative property of the Hyper-Kloosterman sum (see Estermann, [2], p. 86) we have

$$S(a_1, \dots, a_{k-1}, a_k; q) = S(a_1, \dots, a_{k-1}, a_{k_1}; q_1) S(a_1, \dots, a_{k-1}, a_{k_2}; q_2).$$

Thus by the inductive assumption

$$\begin{aligned} & | S(a_1, \dots, a_{k-1}, a_k; q) | \\ & \leq k^{v(q_1)} (q_1)^{\frac{k-1}{2}} (a_1, a_{k_1}, q_1)^{1/2} \dots (a_{k-1}, a_{k_1}, q_1)^{1/2} \\ & \quad \cdot k^{v(q_2)} (q_2)^{\frac{k-1}{2}} (a_1, a_{k_2}, q_2)^{1/2} \dots (a_{k-1}, a_{k_2}, q_2)^{1/2}. \end{aligned}$$

Since it is easily seen

$$\begin{aligned} (a_1, a_{k_1}, q_1) (a_1, a_{k_2}, q_2) &= (a_1, a_k, q) \\ &\vdots \\ (a_{k-1}, a_{k_1}, q_1) (a_{k-1}, a_{k_2}, q_2) &= (a_{k-1}, a_k, q) \end{aligned}$$

the theorem is proved.

Theorem 2 is proved similarly.

Note. By symmetry, the $(a_1, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}$ term in Theorems 1 and 2 may be replaced by

$$\begin{aligned} \min \{ & (a_1, a_k, q)^{1/2} (a_2, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}, \\ & (a_1, a_{k-1}, q)^{1/2} (a_2, a_{k-1}, q)^{1/2} \dots (a_k, a_{k-1}, q)^{1/2}, \\ & \vdots \\ & (a_2, a_1, q)^{1/2} (a_3, a_1, q)^{1/2} \dots (a_k, a_1, q)^{1/2} \}. \end{aligned}$$

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