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# THE HYPER-KLOOSTERMAN SUM

by Lenard WEINSTEIN

## 1. INTRODUCTION

Deligne, [1], has recently proved the very deep theorem on the bound of the Hyper-Kloosterman sum. His estimate results from his solutions of the strong forms of the Weil conjectures.

The Hyper-Kloosterman sum is defined:

$$S(a_1, \dots, a_k; p) = \sum e\left(\frac{a_1 x_1 + \dots + a_k x_k}{p}\right)$$

where  $a_1, \dots, a_k, \alpha$  are non-zero elements of the odd prime field  $F_p$ , and the summation runs through the  $k$  variables  $x_i \in F_p$  with the relation  $\prod x_i = \alpha$ .

Deligne has shown:

$$|S(a_1, \dots, a_k; p)| \leq k p^{\frac{k-1}{2}}.$$

Here, we prove the following generalization for the bound of the Hyper-Kloosterman sum. Define:

$$S(a_1, \dots, a_k; q) = \sum e\left(\frac{a_1 x_1 + \dots + a_k x_k}{q}\right),$$

where  $a_1, \dots, a_k$  are arbitrary integers,  $q$  a positive integer, and the summation runs through the  $k$  variables  $x_i, 0 < x_i \leq q, x_i$  relatively prime to  $q$ , with the relation  $\prod x_i \equiv 1 \pmod{q}$ .

We show:

**THEOREM 1.** *Let  $q$  be an odd positive integer. Then:*

$$|S(a_1, \dots, a_k; q)| \leq k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{\frac{1}{2}} \dots (a_{k-1}, a_k, q)^{\frac{1}{2}}$$

where  $v(q)$  is the number of different prime factors of  $q$ .

THEOREM 2. Let  $q$  be an even positive integer. Then :

$$|S(a_1, \dots, a_k; q)| \leq 2^{\frac{k+1}{2}} k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{\frac{1}{2}} \dots (a_{k-1}, a_k, q)^{\frac{1}{2}}.$$

Estermann, [2], has dealt with the case of the Kloosterman sum.

## 2. LEMMAS

*Lemma 1.* Consider the congruence:

$$x^k \equiv a \pmod{p^m}$$

where  $k, m$  are positive integers,  $a$  is an integer,  $p$  a prime and  $(a, p) = 1$ . Then:

1. If  $p > 2$ , this congruence has at most  $k$  incongruent solutions mod  $p^m$ .
2. If  $p = 2$  and  $k$  is odd, then this congruence has exactly 1 solution mod  $p^m$ .
3. If  $p = 2$ , and  $k = 2^r l$ ,  $r > 1$ ,  $l$  odd, then this congruence has at most  $\min\{2^{r+1}, p^m\}$  solutions mod  $p^m$ .

*Proof:* This is essentially found on pp. 115, 119 of [3].

*Lemma 2.* Let  $p$  be a prime, and  $m, n$  positive integers,  $\frac{1}{2}m \leq n < m$ . Let  $y_1, \dots, y_{k-1}, z_1, \dots, z_{k-1}$  be integers;  $p \nmid y_1, \dots, p \nmid y_{k-1}$ . Define  $[y_1, \dots, y_{k-1}; p^m]$  as that integer  $y$ ,  $0 < y < p^m$  such that  $y(y_1 \dots y_{k-1}) \equiv 1 \pmod{p^m}$ . Then:

$$\begin{aligned} [y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}; p^m] &\equiv [y_1, \dots, y_{k-1}; p^m] \\ &\quad - [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m] p^n z_1 \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad - [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^n z_{k-1} \pmod{p^m} \end{aligned}$$

*Proof:* This follows from the relation

$$[y_1; p^m] \dots [y_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^m] \pmod{p^m}$$

and Lemma 1 of [2].