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# THE HYPER-KLOOSTERMAN SUM

by Lenard WEINSTEIN

## 1. INTRODUCTION

Deligne, [1], has recently proved the very deep theorem on the bound of the Hyper-Kloosterman sum. His estimate results from his solutions of the strong forms of the Weil conjectures.

The Hyper-Kloosterman sum is defined:

$$S(a_1, \dots, a_k; p) = \sum e\left(\frac{a_1 x_1 + \dots + a_k x_k}{p}\right)$$

where  $a_1, \dots, a_k, \alpha$  are non-zero elements of the odd prime field  $F_p$ , and the summation runs through the  $k$  variables  $x_i \in F_p$  with the relation  $\prod x_i = \alpha$ .

Deligne has shown:

$$|S(a_1, \dots, a_k; p)| \leq k p^{\frac{k-1}{2}}.$$

Here, we prove the following generalization for the bound of the Hyper-Kloosterman sum. Define:

$$S(a_1, \dots, a_k; q) = \sum e\left(\frac{a_1 x_1 + \dots + a_k x_k}{q}\right),$$

where  $a_1, \dots, a_k$  are arbitrary integers,  $q$  a positive integer, and the summation runs through the  $k$  variables  $x_i, 0 < x_i \leq q, x_i$  relatively prime to  $q$ , with the relation  $\prod x_i \equiv 1 \pmod{q}$ .

We show:

**THEOREM 1.** *Let  $q$  be an odd positive integer. Then:*

$$|S(a_1, \dots, a_k; q)| \leq k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{\frac{1}{2}} \dots (a_{k-1}, a_k, q)^{\frac{1}{2}}$$

where  $v(q)$  is the number of different prime factors of  $q$ .

THEOREM 2. Let  $q$  be an even positive integer. Then :

$$|S(a_1, \dots, a_k; q)| \leq 2^{\frac{k+1}{2}} k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{\frac{1}{2}} \dots (a_{k-1}, a_k, q)^{\frac{1}{2}}.$$

Estermann, [2], has dealt with the case of the Kloosterman sum.

## 2. LEMMAS

*Lemma 1.* Consider the congruence:

$$x^k \equiv a \pmod{p^m}$$

where  $k, m$  are positive integers,  $a$  is an integer,  $p$  a prime and  $(a, p) = 1$ . Then:

1. If  $p > 2$ , this congruence has at most  $k$  incongruent solutions mod  $p^m$ .
2. If  $p = 2$  and  $k$  is odd, then this congruence has exactly 1 solution mod  $p^m$ .
3. If  $p = 2$ , and  $k = 2^r l$ ,  $r > 1$ ,  $l$  odd, then this congruence has at most  $\min\{2^{r+1}, p^m\}$  solutions mod  $p^m$ .

*Proof:* This is essentially found on pp. 115, 119 of [3].

*Lemma 2.* Let  $p$  be a prime, and  $m, n$  positive integers,  $\frac{1}{2}m \leq n < m$ . Let  $y_1, \dots, y_{k-1}, z_1, \dots, z_{k-1}$  be integers;  $p \nmid y_1, \dots, p \nmid y_{k-1}$ . Define  $[y_1, \dots, y_{k-1}; p^m]$  as that integer  $y$ ,  $0 < y < p^m$  such that  $y(y_1 \dots y_{k-1}) \equiv 1 \pmod{p^m}$ . Then:

$$\begin{aligned} [y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}; p^m] &\equiv [y_1, \dots, y_{k-1}; p^m] \\ &\quad - [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m] p^n z_1 \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad - [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^n z_{k-1} \pmod{p^m} \end{aligned}$$

*Proof:* This follows from the relation

$$[y_1; p^m] \dots [y_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^m] \pmod{p^m}$$

and Lemma 1 of [2].

*Lemma 3.* Let  $p$  be a prime,  $m, n$  positive integers,  $m = 2n + 1$ . Let  $y_1, \dots, y_{k-1}, z_1, \dots, z_{k-1}$  be integers;  $p \nmid y_1, \dots, p \nmid y_{k-1}$ . Then

$$\begin{aligned}
 & [y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^m] \\
 & + [y_1; p^m]^3 [y_2; p^m] \dots [y_{k-1}; p^m] p^{2n} z_1^2 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & + [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^3 p^{2n} z_{k-1}^2 \\
 & - [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m] p^n z_1 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & - [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^n z_{k-1} \\
 & + [y_1; p^m]^2 [y_2; p^m]^2 [y_3; p^m] \dots [y_{k-1}; p^m] p^{2n} z_1 z_2 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & + [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^{2n} z_1 z_{k-1} \\
 & + [y_1; p^m] [y_2; p^m]^2 [y_3; p^m]^2 [y_4; p^m] \dots [y_{k-1}; p^m] p^{2n} z_2 z_3 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & + [y_1; p^m] [y_2; p^m]^2 [y_3; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^{2n} z_2 z_{k-1} \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & + [y_1; p^m] \dots [y_{k-3}; p^m] [y_{k-2}; p^m]^2 [y_{k-1}; p^m]^2 p^{2n} z_{k-2} z_{k-1} \\
 & \pmod{p^m}
 \end{aligned}$$

*Proof:* This follows from Lemma 5 of [2].

*Lemma 4.* Let  $p > 2$  be a prime, and  $n$  a positive integer. Let  $a, h$  be integers. Then:

$$\left| \sum_{0 \leq z < p^{n+1}} e(az^2 p^{-1} + hzp^{-n-1}) \right| = \begin{cases} 0 & p^n \nmid h \\ p^{n+\frac{1}{2}} & p^n \mid h, \quad p \nmid a \\ p^{n+1} & p^{n+1} \mid h, \quad p \mid a \\ 0 & p^{n+1} \nmid h, \quad p \mid a. \end{cases}$$

*Proof:* The first two parts of this lemma are Lemma 5 of [2]. The last two parts are trivial.

### 3. PROOF OF THEOREMS 1 AND 2

PROPOSITION 1. Let  $p$  be a prime,  $m$  a positive integer and  $a_1, \dots, a_k$ , integers such that

$$(a_1, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = p^h \quad 0 \leq h < m.$$

Then

$$S(a_1, \dots, a_k; p^m) = (p^h)^{k-1} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h})$$

*Proof:* The proof is similar to that of [2], page 85 bottom.

PROPOSITION 2. Let  $m, n$  be positive integers  $\frac{1}{2}m \leq n < m$ ,  $p$  a prime, and  $a_1, \dots, a_k$  integers such that  $(a_1, a_k; p^m) = 1$ . Then:

$$|S(a_1, \dots, a_k; p^m)| \leq A(p^n)^{k-1}$$

where

$$A = \begin{cases} k & \text{if } p > 2. \\ 1 & \text{if } p = 2 \text{ and } k \text{ is odd.} \\ \min \{ 2^{r+1}, p^m \} & \text{if } p = 2 \text{ and } k = 2^r l, \\ & r > 1 \text{ and } l \text{ odd.} \end{cases}$$

*Proof:* Let us assume throughout this proposition that  $S(a_1, \dots, a_k; p^m) \neq 0$ , or else we are done.

Now we have the identity

$$\sum_{\substack{0 < x_1, \dots, x_{k-1} \leq p^m \\ p \nmid x_1, \dots, p \nmid x_{k-1}}} f(x_1, \dots, x_{k-1})$$

$$= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^{m-n}} f(y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}).$$

Letting

$$f(x_1, \dots, x_{k-1}) = e\left(\frac{a_1 x_1 + \dots + a_{k-1} x_{k-1} + a_k [x_1, \dots, x_{k-1}; p^m]}{p^m}\right)$$

we see, using Lemma 2

$$= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} S(a_1, \dots, a_k; p^m) e\left(\frac{a_1 y_1 + \dots + a_{k-1} y_{k-1} + a_k [y_1, \dots, y_{k-1}; p^m]}{p^m}\right)$$

$$\sum_{0 \leq z_1 < p^{m-n}} e(\{a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m]\} p^{n-m} z_1)$$

$$\vdots$$

$$\sum_{0 \leq z_{k-1} < p^{m-n}} e(\{a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2\} p^{n-m} z_{k-1}).$$

Now since we have assumed  $S(a_1, \dots, a_k; p^m) \neq 0$ , the inner sums above must not equal 0. Thus

$$p^{m-n} \mid a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m]$$

$$\vdots$$

$$p^{m-n} \mid a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2.$$

These congruences imply, since  $(a_1, a_k, p^m) = 1$ , also  $(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = 1$ , and moreover

$$p \nmid a_1, \dots, p \nmid a_k.$$

Now we have

$$\begin{aligned}
& \leq (p^{m-n})^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} |S(a_1, \dots, a_k; p^m)| \\
& \quad a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{m-n}} \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^{m-n}} \\
& = (p^n)^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-n} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} |S(a_1, \dots, a_k; p^m)| \\
& \quad a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{m-n}} \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^{m-n}}
\end{aligned}$$

Now the congruences in the above sum are easily seen to be equivalent to :

$$\begin{aligned}
a_1 y_1 &\equiv a_2 y_2 \equiv \dots \equiv a_{k-1} y_{k-1} \pmod{p^{m-n}} \\
y_1^k &\equiv [a_1; p^m]^{k-1} a_2 \dots a_k \pmod{p^{m-n}}.
\end{aligned}$$

Thus by Lemma 1, the proposition is proved.

**PROPOSITION 3.** *Let  $p > 2$  be a prime,  $a_1, \dots, a_k$  integers such that  $(a_1, a_k, p^m) = 1$ , where  $m$  is a positive even integer. Then*

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}}$$

*Proof:* This is Proposition 2, with  $n = \frac{m}{2}$ .

PROPOSITION 4. Let  $p = 2$ ,  $m$  a positive integer,  $a_1, \dots, a_k$  integers such that  $(a_1, a_k, p^m) = 1$ . Then:

$$|S(a_1, \dots, a_k; p^m)| \leq A_1 (p^m)^{\frac{k-1}{2}}$$

where

$$A_1 = \begin{cases} 1 & , \text{ if } m \text{ even, } k \text{ odd.} \\ \min \{ 2^{r+1}, p^m \} & , \text{ if } m \text{ even, } k = 2^r l, r > 1, l \text{ odd.} \\ 2^{\frac{k-1}{2}} & , \text{ if } m \text{ odd, } k \text{ odd.} \\ 2^{\frac{k-1}{2}} \min \{ 2^{r+1}, p^m \} & , \text{ if } m \text{ odd, } k = 2^r l, r > 1, l \text{ odd.} \end{cases}$$

*Proof:* This follows from Proposition 2 with  $n = m - [\frac{1}{2}m]$ .

PROPOSITION 5. Let  $p > 2$  be a prime,  $a_1, \dots, a_k$  integers. Then

$$|S(a_1, \dots, a_k; p)| \leq k p^{\frac{k-1}{2}} (a_1, a_k; p)^{1/2} \dots (a_{k-1}, a_k, p)^{1/2}$$

*Proof:* If  $p \nmid a_1 \dots a_k$  this is Deligne's theorem. Therefore suppose, without loss of generality that  $p \mid a_k, \dots, p \mid a_{k-i+1}$  where  $i \geq 1$ . Thus:

$$\begin{aligned} S(a_1, \dots, a_k; p) &= (p-1)^{i-1} \sum_{0 < x_1 < p} e\left(\frac{a_1 x_1}{p}\right) \dots \sum_{0 < x_{k-i} < p} e\left(\frac{a_{k-i} x_{k-i}}{p}\right) \\ &= (p-1)^{i-1} (-1)^{k-i} \end{aligned}$$

and so the proposition is proved.

PROPOSITION 6. Let  $p > 2$  be a prime and  $m > 1$  an odd positive integer. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

*Proof:* Let us assume throughout this proposition that  $S(a_1, \dots, a_k; p^m) \neq 0$ , or else we are done. Let  $\frac{m-1}{2} = n > 0$ .

Now we have the identity:

$$\begin{aligned}
& \sum_{\substack{0 < x_1, \dots, x_{k-1} \leq p^{2n+1} \\ p \nmid x_1, \dots, p \nmid x_{k-1}}} f(x_1, \dots, x_{k-1}) \\
= & \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^{n+1}} f(y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}).
\end{aligned}$$

Letting

$$f(x_1, \dots, x_{k-1}) = e\left(\frac{a_1 x_1 + \dots + a_{k-1} x_{k-1} + a_k [x_1, \dots, x_{k-1}; p^m]}{p^m}\right)$$

we see, using Lemma 3

$$\begin{aligned}
& S(a_1, \dots, a_k; p^m) \\
= & \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} e\left(\frac{a_1 y_1 + \dots + a_{k-1} y_{k-1} + a_k [y_1, \dots, y_{k-1}; p^m]}{p^m}\right) \\
& \sum_{0 \leq z_{k-1} < p^{n+1}} e(\{a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2\} p^{-n-1} z_{k-1} \\
& \quad + [y_1; p^m] \dots [y_{k-1}; p^m]^3 a_k p^{-1} z_{k-1}^2) \\
& \cdot \\
& \cdot \\
& \cdot \\
& \sum_{0 \leq z_1 < p^{n+1}} e(\{a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \\
& \quad + a_k [y_1; p^m]^2 [y_2; p^m]^2 \dots [y_{k-1}; p^m] z_2 p^n \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad + a_k [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m]^2 z_{k-1} p^n\} p^{-n-1} z_1 \\
& \quad + a_k [y_1; p^m]^3 \dots [y_{k-1}; p^m] z_2^1 p^{-1})
\end{aligned}$$

Since  $S(a_1, \dots, a_k, p^m)$  is assumed to be non-zero, we see by Lemma 4 that:

$$\begin{aligned}
 p^n \mid & \{ a_1 - a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \\
 & + a_k [y_1; p^m]^2 [y_2; p^m]^2 \dots [y_{k-1}; p^m] z_2 p^n \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + a_k [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m]^2 z_{k-1} p^n \} \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 p^n \mid & \{ a_{k-1} - a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \} .
 \end{aligned}$$

Now let us assume  $(a_1, a_k, p^m) = 1$ . By reasoning similar to that of Proposition 2, we see that

$$(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = 1 ,$$

and that  $p \nmid a_k$ . Thus by Lemma 4:

$$\begin{aligned}
 & | S(a_1, \dots, a_k; p^m) | \\
 \leq & (p^{n+1/2})^{k-1} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^n \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} 1 \\
 a_1 & \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^n} \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 a_{k-1} & \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m]^2 \pmod{p^n}
 \end{aligned}$$

Now by reasoning as in Proposition 2 we see  $p \nmid a_1, \dots, p \nmid a_{k-1}$ , and so by Lemma 1:

$$| S(a_1, \dots, a_k; p^m) | \leq k p^{(n+1/2)k-1} .$$

Now let us assume

$$(a_1, a_k, p^m) = p^h, \quad 0 < h < n + 1 ,$$

(if this case is possible.)

Thus  $p \mid a_k$ , and Lemma 4 now shows:

$$a_1 \equiv a_k [y_1; p^m]^2 \dots [y_{k-1}; p^m] \pmod{p^{n+1}}$$

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$$a_{k-1} \equiv a_k [y_1; p^m] \dots [y_{k-1}; p^m] \pmod{p^{n+1}}.$$

Thus:

$$(a_2, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = p^h.$$

Thus by Proposition 1, we have:

$$S(a_1, \dots, a_k; p^m) = p^{(k-1)h} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h}).$$

Now by Proposition 3, 5 and the first part of this proposition, we have:

$$|S(a_1, \dots, a_k; p^m)| \leq k p^{(k-1)h} (p^{m-h})^{\frac{k-1}{2}} = k (p^m)^{\frac{k-1}{2}} (p^h)^{\frac{k-1}{2}}.$$

Now let us assume

$$(a_1, a_k, p^m) = p^{h_1}, \quad h_1 \geq n+1.$$

As in the previous argument we see

$$(a_2, a_k, p^m) = p^{h_2}, \quad h_2 \geq n+1$$

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$$(a_{k-1}, a_k, p^m) = p^{h_{k-1}}, \quad h_{k-1} \geq n+1.$$

Let  $h = \min \{h_1, \dots, h_{k-1}\}$ . We may assume  $h < m$  or else the result is trivial. Now

$$\begin{aligned} & S(a_1, \dots, a_k; p^m) \\ &= \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-h} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} \sum_{0 \leq z_1, \dots, z_{k-1} < p^h} e \\ & \left( \frac{a_1 (y_1 + p^{m-h} z_1) + \dots + a_k [y_1 + p^{m-h} z_1, \dots; p^m]}{p^m} \right). \end{aligned}$$

Now since  $p^m \mid a_1 p^{m-h}, \dots, p^m \mid a_{k-1} p^{m-h}$  and since

$$[y_1 + p^{m-h} z_1, \dots, y_{k-1} + p^{m-h} z_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^{m-h}] \pmod{p^{m-h}}$$

we have

$$\begin{aligned}
 & S(a_1, \dots, a_k; p^m) \\
 = & p^{h(k-1)} \sum_{\substack{0 < y_1, \dots, y_{k-1} \leq p^{m-h} \\ p \nmid y_1, \dots, p \nmid y_{k-1}}} e\left(\frac{a_1 p^{-h} y_1 + \dots + a_k p^{-h} [y_1, \dots, y_{k-1}; p^{m-h}]}{p^{m-h}}\right) \\
 & = p^{h(k-1)} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h}).
 \end{aligned}$$

Now we may assume without loss of generality that  $h = h_1$ . Thus by Propositions 3, 5 and the first part of this proposition,

$$\begin{aligned}
 |S(a_1, \dots, a_k; p^m)| & \leq k p^{h(k-1)} p^{(m-h)\binom{k-1}{2}} \\
 & = k (p^m)^{\frac{k-1}{2}} (p^h)^{\frac{k-1}{2}}.
 \end{aligned}$$

PROPOSITION 7. Let  $p > 2$  be a prime,  $m$  an even positive integer, and  $a_1, \dots, a_k$  integers. Then:

$$|S(a_1, \dots, a_k; p^m)| \leq k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

*Proof:* Using the identity of Proposition 2 and the results of Propositions 3, 5, 6, this is proved as Proposition 6.

PROPOSITION 8. Let  $p = 2$ ,  $m$  a positive integer,  $a_1, \dots, a_k$  integers. Then

$$|S(a_1, \dots, a_k; p^m)| \leq 2^{\frac{k+1}{2}} k (p^m)^{\frac{k-1}{2}} (a_1, a_k, p^m)^{1/2} \dots (a_{k-1}, a_k, p^m)^{1/2}.$$

*Proof:* This is proved as Proposition 7.

THEOREM 1. Let  $q$  be a positive odd integer. Then for any integers  $a_1, \dots, a_k$ :

$$|S(a_1, \dots, a_k; q)| \leq k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}.$$

*Proof:* We proceed by induction on  $q$ . For  $q = 1$  the theorem is trivial. Assume the theorem true for all  $S(b_1, \dots, b_k; q')$ ,  $q' < q$ ,  $b_1, \dots, b_k$  integers.

Now consider  $S(a_1, \dots, a_k; q)$ .

By Propositions 5, 6, 7, we may assume  $q$  is not a prime power; hence there exist odd  $q_1, q_2$  such that  $q = q_1 q_2$ ,  $(q_1, q_2) = 1$ ,  $q_1 > 1$ ,  $q_2 > 1$ . Thus there exist integers  $a_{k_1}, a_{k_2}$  such that

$$a_k = a_{k_1} q_2^k + a_{k_2} q_1^k.$$

By the multiplicative property of the Hyper-Kloosterman sum (see Estermann, [2], p. 86) we have

$$S(a_1, \dots, a_{k-1}, a_k; q) = S(a_1, \dots, a_{k-1}, a_{k_1}; q_1) S(a_1, \dots, a_{k-1}, a_{k_2}; q_2).$$

Thus by the inductive assumption

$$\begin{aligned} & | S(a_1, \dots, a_{k-1}, a_k; q) | \\ & \leq k^{v(q_1)} (q_1)^{\frac{k-1}{2}} (a_1, a_{k_1}, q_1)^{1/2} \dots (a_{k-1}, a_{k_1}, q_1)^{1/2} \\ & \quad \cdot k^{v(q_2)} (q_2)^{\frac{k-1}{2}} (a_1, a_{k_2}, q_2)^{1/2} \dots (a_{k-1}, a_{k_2}, q_2)^{1/2}. \end{aligned}$$

Since it is easily seen

$$\begin{aligned} (a_1, a_{k_1}, q_1) (a_1, a_{k_2}, q_2) &= (a_1, a_k, q) \\ &\vdots \\ (a_{k-1}, a_{k_1}, q_1) (a_{k-1}, a_{k_2}, q_2) &= (a_{k-1}, a_k, q) \end{aligned}$$

the theorem is proved.

Theorem 2 is proved similarly.

*Note.* By symmetry, the  $(a_1, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}$  term in Theorems 1 and 2 may be replaced by

$$\begin{aligned} \min \{ & (a_1, a_k, q)^{1/2} (a_2, a_k, q)^{1/2} \dots (a_{k-1}, a_k, q)^{1/2}, \\ & (a_1, a_{k-1}, q)^{1/2} (a_2, a_{k-1}, q)^{1/2} \dots (a_k, a_{k-1}, q)^{1/2}, \\ & \vdots \\ & (a_2, a_1, q)^{1/2} (a_3, a_1, q)^{1/2} \dots (a_k, a_1, q)^{1/2} \}. \end{aligned}$$

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