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# ON COUSIN-*I* COMPLEX SPACES

by Edoardo BALLICO<sup>1)</sup>

## SUMMARY

We study the properties of a complex space such that every closed analytic subspace has the 1-Cousin property. Under mild hypotheses we prove that  $X$  is a Stein space.

G. Berg in [2] proved very easily the following result: Let  $X$  be an open subset of a two-dimensional Stein space  $Y$ . If in  $X$  every 1-Cousin problem is solvable, then  $X$  is a domain of holomorphy in  $Y$ . It is not difficult to modify his proof and obtain that, under the stated assumption,  $X$  is always a Stein space.

In this note we want to consider this problem also for higher dimensional complex spaces. The proofs are always by induction on the dimension; they are similar to the proofs in the two dimensional case. This is also a partial generalization of my previous [1], theorem 2. We say that a complex space  $X$  is Cousin-*I* or has the 1-Cousin property if every 1-Cousin problem on  $X$  is solvable. Any Stein space is a Cousin-*I* space. We denote by  $O(X)$  the algebra of holomorphic functions on a complex space  $X$ . For each  $f \in O(X)$  we put

$$\{x \in X : f(x) = 0\} =: V(f),$$

an analytic subset of  $X$ . In this paper we consider for simplicity only complex spaces with bounded Zariski tangent dimension.

LEMMA 1. *Let  $X$  be a complex space with the 1-Cousin property and  $f \in O(X)$  such that for each  $x \in X$  the germ of  $f$  around  $x$  is a non-zero divisor in  $\mathcal{O}_{X,x}$ . We put*

$$Z := (V(f), \mathcal{O}_Z)$$

with

$$\mathcal{O}_Z := \mathcal{O}_X/f \mathcal{O}_{X|V(f)}.$$

*Then the natural map  $p: O(X) \rightarrow O(Z)$  is surjective.*

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*Proof:* Let  $\{U_i\}$  be a family of Stein open subsets of  $X$  which covers  $V(f)$  and such that for each index  $i$  we have  $U_i \cap V(f) \neq \emptyset$ .

We consider the exact sequence of sheaves on  $X$

$$0 \rightarrow \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0 \quad (1)$$

where  $t$  is the multiplication by  $f$ . It is exact because  $f$  is a non zero-divisor in  $\mathcal{O}_{X,x}$  for each  $x \in X$ . We take  $h \in \mathcal{O}(Z)$ . From theorem B of Cartan-Serre it follows that for every  $i$  the restriction of  $h$  to  $V(f) \cap U_i$  can be extended to a holomorphic function  $f_i \in \mathcal{O}(U_i)$ . If we have  $U_i \cap U_j \neq \emptyset$ , from the restriction of the exact sequence (1) to  $U_i \cap U_j$ , we obtain that  $f_i - f_j = g_{ij} f$ , where  $g_{ij} \in \mathcal{O}(U_i \cap U_j)$  is uniquely determined because  $f$  is not a zero-divisor of  $\mathcal{O}(U_i \cap U_j)$ .

We take the covering of  $X$  given by  $X \setminus V(f)$  and  $\{U_i\}$ . As data for the 1-Cousin problem in  $X$  we take the constant 1 on  $X \setminus V(f)$  and the meromorphic function  $f_i/f$  on  $X$ . Again  $f_i/f$  is a meromorphic function on  $U_i$ , because  $f$  is not a zero-divisor on  $\mathcal{O}_{X,x}$  for each  $x \in U_i$ .

We obtain a meromorphic function  $g$  on  $X$  which satisfies the thesis of the 1-Cousin problem. The function  $gf$  is holomorphic on  $X$  and gives an extension of  $h$  to  $X$ .  $\square$

G. Berg in [2] lemma, gave a proof more or less of the lemma above, if  $V(f)$  is a Stein space. He uses a strong theorem of Siu about the existence of a Stein open neighborhood of any Stein analytic subspace in any complex space. Our proof is similar, but does not use this strong theorem and is therefore more elementary.

**THEOREM 1.** *Let  $X$  be a complex  $n$ -dimensional space, such that  $\mathcal{O}(X)$  gives local coordinates (in particular it is holomorphically spreadable). Suppose that  $X$  and every closed analytic subspace have the 1-Cousin property. Suppose that  $X$  is either reduced or Cohen-Macaulay. Then  $X$  is a Stein space.*

*Proof:* The proof is by induction on  $n = \dim X$ . If we have  $n \leq 1$ , then  $X$  is a Stein space because it does not contain any positive dimensional analytic compact subspace. Suppose  $n \geq 2$ .

We put as in [6]  $S_d = \{x \in X : \text{prof}_x(\mathcal{O}_{X,x}) \leq d\}$ . By [6] theorem 1.11,  $S_d$  is an analytic closed subspace of  $X$  with  $\dim S_d \leq d$ . For each  $f \in \mathcal{O}(X)$ , we have  $\dim(V(f) \cap S_{k+1}) \leq k$  for each integer  $k$  if and only if  $f_x$  is a non zero-divisor in  $\mathcal{O}_{X,x}$  for each  $x \in X$  by [6], Corollary 1.18.

We may assume that  $X$  has no isolated point. Therefore, both under the hypotheses that  $X$  is reduced or that  $X$  is Cohen-Macaulay, we obtain  $S_0 = \emptyset$ .

By a simple application of Baire's category theorem, we obtain  $f \in O(X)$  such that  $h$  has two different values on each irreducible component of dimension  $k$  of  $S_k$ .

First we want to prove that  $X$  is holomorphically separated. We fix two distinct points  $x, y \in X$ . If we have  $f(x) \neq f(y)$ , the result is trivially true. Suppose  $f(x) = f(y)$ . By considering instead of  $f$  the function  $f - f(x)$ , we may suppose that  $f$  vanishes at  $x$ . Therefore we suppose  $x, y \in V(f)$  and we put  $Z := (V(f), \mathcal{O}_Z)$  with

$$\mathcal{O}_Z := \mathcal{O}_X/f \mathcal{O}_{X|V(f)}.$$

a) If  $X$  is reduced,  $V(f)$  with the reduced structure is by the inductive hypothesis a Stein space. But a complex space  $Y$  is a Stein space if and only if  $Y_{\text{red}}$  is a Stein space. Therefore  $Z$  is a Stein space, too.

b) If  $X$  is Cohen-Macaulay, then also  $Z$  is Cohen-Macaulay. In fact for each  $x \in X$  with  $f(x) = 0$ , we have

$$\text{prof}(\mathcal{O}_{X,x}/f \mathcal{O}_{X,x}) = \text{prof}(\mathcal{O}_{X,x}) - 1$$

by [6], lemma 1.2, and

$$\dim(\mathcal{O}_{X,x}/f \mathcal{O}_{X,x}) = \dim \mathcal{O}_{X,x} - 1.$$

From the inductive hypothesis it follows that  $Z$  is a Stein space.

Therefore under both assumptions we have proved that  $Z$  is a Stein space. In particular there exists a  $g \in O(Z)$  such that  $g(x) \neq g(y)$ . From lemma 1 it follows that there exists  $G \in O(X)$  which extends  $g$ . In particular we have  $G(x) \neq G(y)$ , proving that  $X$  is holomorphically separated.

Let  $\chi$  be a character of  $O(X)$ ;  $\chi$  is a multiplicative functional from  $O(X)$  to  $\mathbf{C}$ . By [5], Corollary pag. 222, or [4], pag. 182, to prove that  $X$  is a Stein space it is sufficient to demonstrate that  $\chi$  is a valuation in a point of  $X$ . We take  $h \in O(X)$  as before, i.e. such that  $h$  has two different values on each irreducible component of dimension  $k$  of  $S_k$ . We put  $f = h - \chi(h)$ . We put  $H := \ker(\chi)$ .  $H$  is a maximal ideal of  $O(X)$ . The function  $f$  is in  $H$ . We define  $Z$  as above.  $Z$  is again a Stein space by the inductive hypothesis. From lemma 1 and the exact sequence (1) in lemma 1, we obtain an exact sequence

$$0 \rightarrow f O(X) \rightarrow O(X) \rightarrow O(Z) \rightarrow 0 \tag{2}$$

Since  $f \in \ker(\chi)$ ,  $\chi$  induces a character  $\chi'$  on  $O(Z)$ . Since  $Z$  is a Stein space,  $\chi'$  is induced by the valuation in a point  $x \in Z \subset X$ . If  $\{g_i\} \in O(Z)$  generate  $\ker(\chi')$

and  $p(f_i) = g_i$ , the functions  $f_i$  and  $f$  generate  $H = \ker(\chi)$ . Therefore, for every  $m \in H$  we have  $m(x) = 0$ . Since  $M$  is a maximal ideal,  $\chi$  is the valuation at the point  $x$ .  $\square$

If  $\dim X = 2$ , then it is sufficient to assume the 1-Cousin property for  $X$ . In fact its 1-dimensional subspace  $Z$  in the proof above is a Stein space because it has no compact, analytic subspace of positive dimension. If  $X$  is Cohen-Macaulay, it follows from lemma 1 that it is sufficient to assume that  $X$  and every closed analytic subspace  $Z$  of  $X$  given by global equations  $f_1, \dots, f_k$  and with

$$\mathcal{O}_Z := \mathcal{O}_X/(f_1, \dots, f_k) \mathcal{O}_{X|Z}$$

are Cousin-*I* space.

If in the theorem above we omit the condition that  $X$  has local coordinates given by global functions, we obtain only that every character is given by a valuation at a point of  $X$ . An easy inductive argument shows also that for every character  $\chi$  of  $\mathcal{O}(X)$ ,  $\ker(\chi)$  is finitely generated. Therefore we obtain the following

**PROPOSITION 1.** *Let  $X$  be a holomorphically spreadable complex space such that every closed analytic subspace has the 1-Cousin property. Suppose that  $X$  is reduced or Cohen-Macaulay. Then*

- a)  *$X$  is holomorphically separated,*
- b) *every character  $\chi$  of  $\mathcal{O}(X)$  is a valuation at a point of  $X$  and  $\ker(\chi)$  is finitely generated.*

**THEOREM 2.** *Let  $X$  be a complex reduced space which is a relatively compact open subset of a holomorphically separated, reduced complex space  $Y$ . Then  $X$  is a Stein space if and only if any closed analytic subspace, with its reduced structure, has the 1-Cousin property.*

*Proof :* The “only if” part is well-known. We have  $\dim X = n < +\infty$  and we use induction on  $n$ . For  $n \leq 1$  the result is well-known. Now suppose  $n \geq 2$ . Let  $\{x_n\}$  be a sequence in  $X$  without accumulation points in  $X$ . We may suppose, eventually extracting a subsequence, that  $\{x_n\}$  converges to a point  $y \in \mathcal{O}(X)$ . If we take into account the proof of [3], theorem 1.3, it remains only to prove that there exist

$$f_1, \dots, f_k \in \mathcal{O}(Y)$$

with only  $y$  as a common zero and such that there exist

$$g_1, \dots, g_k \in \mathcal{O}(X)$$

with  $f_1g_1 + \dots + f_kg_k = 1$  on  $X$ . In fact one of the  $g_i$ 's must be unbounded on the sequence  $\{x_n\}$ . As in the proof of theorem 1 we take  $f \in \mathcal{O}(Y)$  a non zero-divisor in  $\mathcal{O}_{Y,x}$  for each  $x \in Y$  and vanishing in the point  $y$ . By the inductive hypothesis  $V(f) \cap X$  with its reduced structure is a Stein space. We define  $W := (V(f), \mathcal{O}_W)$  with

$$\mathcal{O}_W := \mathcal{O}_Y/f \mathcal{O}_{Y|V(f)}$$

and  $Z := W|_{X \wedge W}$  is a Stein space, too. Let  $p: \mathcal{O}(X) \rightarrow \mathcal{O}(Z)$  be the projection. From lemma 1 it follows that  $p$  is surjective. Let  $h_1, \dots, h_k$  be holomorphic functions on  $W$  such that  $y$  is their unique common zero. Since  $Z$  is a Stein space and  $h_1, \dots, h_k$  have no common zero in  $Z$ , there exist

$$g_1, \dots, g_k \in \mathcal{O}(Z)$$

such that we have  $g_1h_1|_Z + \dots + g_kh_k|_Z = 1$  on  $Z$ . We take  $G_i \in \mathcal{O}(X)$  such that  $p(G_i)$  is  $g_i$ . By the exact sequence (2) we obtain that there exists  $G \in \mathcal{O}(X)$  such that we have  $Gf + \dots + G_kh_k = 1$  on  $X$ . The proof is now finished.  $\square$

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