

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 27 (1981)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** APPENDIX: TORSION POINTS OF ABELIAN VARIETIES IN CYCLOTOMIC EXTENSIONS  
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**DOI:** <https://doi.org/10.5169/seals-51754>

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# APPENDIX: TORSION POINTS OF ABELIAN VARIETIES IN CYCLOTOMIC EXTENSIONS

by Kenneth A. RIBET <sup>1)</sup>

Let  $k$  be a number field, and let  $\bar{k}$  be an algebraic closure for  $k$ . For each prime  $p$ , let  $K_p$  be the subfield of  $k$  obtained by adjoining to  $k$  all  $p$ -power roots of unity in  $\bar{k}$ . Let  $K$  be the compositum of all of the  $K_p$ , i.e., the field obtained by adjoining to  $k$  all roots of unity in  $\bar{k}$ .

Suppose that  $A$  is an abelian variety over  $k$ . Mazur has raised the question of whether the groups  $A(K_p)$  are finitely generated [4]. In this connection, H. Imai [1] and J.-P. Serre [5] proved (independently) that the *torsion subgroup* of  $A(K_p)$  is finite for each  $p$ . The aim of this appendix is to prove that more precisely one has the following theorem, cf. [3], §II, Remark 3.

**THEOREM 1.** *The torsion subgroup  $A(K)_{\text{tors}}$  of  $A(K)$  is finite.*

Let  $G$  be the Galois group  $\text{Gal}(\bar{k}/k)$  and let  $H$  be its subgroup  $\text{Gal}(\bar{k}/K)$ . For each positive integer  $n$ , let  $A[n]$  be the kernel of multiplication by  $n$  in  $A(\bar{k})$ . For each prime  $p$ , let  $V_p$  be the  $\mathbf{Q}_p$ -adic Tate module attached to  $A$ . If  $M$  is one of these modules, we denote by  $M^H$  the set of elements of  $M$  left fixed by  $H$ . Since  $H$  is normal in  $G$ ,  $M^H$  is stable under the action of  $G$  on  $M$ .

Because of the structure of the torsion subgroup of  $A(\bar{k})$ , one sees easily that Theorem 1 is equivalent to the conjunction of the following two statements:

**THEOREM 2.** *For all but finitely many primes  $p$ , we have  $A[p]^H = 0$ .*

**THEOREM 3.** *For each prime  $p$ , we have  $V_p^H = 0$ .*

Indeed, Theorem 2 asserts the vanishing of the  $p$ -primary part of  $A(K)_{\text{tors}}$ , while Theorem 3 asserts the finiteness of this  $p$ -primary part.

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<sup>1)</sup> Partially supported by National Science Foundation contract number MCS 80-02317.

In proving these statements, we visibly have the right to replace  $k$  by a finite extension of  $k$ . Therefore, using ([SGA 71], IX, 3.6) we can (and will) assume that  $A/k$  is semistable. Next, consider the largest subextension  $k'$  of  $K/k$  which is unramified at all finite places of  $k$ .

**LEMMA.** *For each prime  $p$ , let  $L_p$  be the largest extension of  $k$  in  $K$  which is unramified at all places of  $k$  except for primes dividing  $p$  and the infinite places of  $k$ . Then  $L_p$  is the compositum  $k'K_p$ .*

*Proof.* Let  $A$  be the Galois group  $\text{Gal}(K/k)$ , viewed as a subgroup of  $\hat{\mathbf{Z}}^*$ . We consider  $\hat{\mathbf{Z}}^*$  as the direct product of its two subgroups  $\mathbf{Z}_p^*$  and  $\prod_{l \neq p} \mathbf{Z}_l^*$ . Let  $I$  (resp.  $J$ ) be the subgroup of  $A$  generated by the inertia groups of  $A$  for primes of  $k$  which divide  $p$  (resp. which do not divide  $p$ ). Then  $I$  is a subgroup of  $\mathbf{Z}_p^*$ , while  $J$  is a subgroup of  $\prod_{l \neq p} \mathbf{Z}_l^*$ . The product  $I \times J$  is the subgroup of  $A$  generated by all inertia groups of  $A$ . We have  $J = \text{Gal}(\bar{k}/L_p)$ ,  $I \times J = \text{Gal}(\bar{k}/k')$ , and  $\text{Gal}(\bar{k}/K_p) = A \cap \left(\prod_{l \neq p} \mathbf{Z}_l^*\right)$ . Now  $\text{Gal}(\bar{k}/k'K_p)$  is the intersection of the two Galois groups  $\text{Gal}(\bar{k}/k')$  and  $\text{Gal}(\bar{k}/K_p)$ . Putting these facts together, we prove the desired assertion.

We now replace  $k$  by its finite extension  $k'$ . With this replacement made,  $K_p$  becomes equal to  $L_p$ . Furthermore, for odd primes  $p$ , the largest extension of  $k$  in  $K$  which is unramified outside  $p$  and infinity and which has degree prime to  $p$  is the field obtained by adjoining to  $k$  the  $p$ -th roots of unity in  $\bar{k}$ .

*Proof of Theorem 2.* We shall consider only primes  $p$  which are odd, unramified in  $k$ , and such that  $A$  has good reduction at at least one prime of  $k$  dividing  $p$ . Let  $p$  be such a prime and  $v$  a prime of  $k$  over  $p$  at which  $A$  has good reduction. Suppose that the  $G$ -module  $A[p]^H$  is non-zero, and let  $W$  be a simple  $G$ -submodule of this module. The algebra  $\text{End}_G W$  is a finite field  $\mathbf{F}$ , and the action of  $G$  on  $W$  is given by a character

$$\phi: G \rightarrow \mathbf{F}^*$$

since the action of  $G$  on  $A[p]^H$  is abelian. (Here the point is simply that  $G/H$  is an abelian group.) In particular, the image of  $G$  in  $\text{Aut}(A[p])$  has order prime to  $p$ . On the other hand, the character  $\phi$  is unramified at primes of  $k$  not dividing  $p$  because  $A/k$  is semistable. By the discussion following the lemma, we know that  $\phi$  factors through the quotient  $\text{Gal}(k(\mu_p)/k)$  of  $G$ ; here,  $\mu_p$  denotes the group of  $p$ -th roots of unity. In particular,  $\phi$  must have order dividing  $p - 1$ , so that its

values lie in the prime field  $\mathbf{F}_p$ . Since  $W$  was chosen to be simple, its dimension over  $\mathbf{F}_p$  must be 1; i.e.,  $W$  is a group of order  $p$ .

Let  $\chi: G \rightarrow \mathbf{F}_p^*$  be the mod  $p$  cyclotomic character, i.e., the character giving the action of  $G$  on  $\mu_p$ . Since  $\phi$  factors through  $\text{Gal}(k(\mu_p)/k)$ , we may write  $\phi$  in the form  $\chi^n$ , where  $n$  is an integer mod  $(p-1)$ . We claim that  $n$  can only be 0 or 1.

To verify this claim, it is enough to check that it is true after we replace  $G$  by an inertia group  $I$  in  $G$  for the prime  $v$ , since  $\chi$  is totally ramified at  $v$ . We remark that  $W$  is the  $I$ -module associated to a finite flat commutative group scheme  $\mathcal{W}$  over the ring of integers of the completion of  $k$  at  $v$ , since  $v$  is such that  $A$  has good reduction at  $v$ . Because  $\mathcal{W}$  has order  $p$ , the classification of Tate-Oort ([8], especially pp. 15-16) applies to  $\mathcal{W}$ . Because  $v$  is absolutely unramified, the classification shows immediately that  $\mathcal{W}$  is either étale or the dual of an étale group. In the former case,  $I$  acts trivially on  $W$ ; in the latter case,  $I$  acts on  $W$  via  $\chi$ . This completes the verification of the claim.

Thus, if Theorem 2 is false, there are infinitely many primes  $p$  for which  $A[p]$  contains a  $G$ -submodule isomorphic to either  $\mathbf{Z}/p\mathbf{Z}$  or to  $\mu_p$ . Of course, the former case can occur only a finite number of times, since  $A(k)$  is finite. One way to rule out the latter case is to argue that whenever  $\mu_p$  is a submodule of  $A[p]$ , the group  $\mathbf{Z}/p\mathbf{Z}$  is a quotient of the dual of  $A[p]$ , which is the kernel of multiplication by  $p$  on the abelian variety  $A^\vee$  dual to  $A$ . In other words, if  $\mu_p$  occurs as a submodule of  $A[p]$ , then there is an abelian variety isogenous to  $A^\vee$  (and therefore in fact to  $A$ ) which has a rational point of order  $p$  over  $k$ . Therefore  $p$  is a divisor of the order of a finite group that may be specified in advance, viz. the group of rational points of any reduction of  $A$  at a good unramified prime of  $k$  of residue characteristic different from 2. (See the appendix to Katz's recent paper [2] for a discussion of this point.)

*Proof of Theorem 3.* Suppose that  $p$  is a prime such that  $V_p^H$  is non-zero. We again choose  $W$  to be an irreducible  $G$ -submodule (i.e.,  $\mathbf{Q}_p[G]$ -submodule) of  $V_p^H$ . Because the action of  $G$  on  $W$  is abelian, and because  $W$  is simple, each element of  $G$  acts semisimply on  $W$ . Since  $A/k$  is semistable, it follows that the homomorphism

$$\rho: G \rightarrow \text{Aut}(W)$$

giving the action of  $G$  on  $W$  is unramified at all primes of  $k$  not dividing  $p$ . Therefore,  $\rho$  factors through  $\text{Gal}(K_p/k)$  in view of the lemma and the subsequent replacement  $k \rightarrow k'$ . In other words, starting from the hypothesis that the  $p$ -torsion subgroup of  $A(K)$  is infinite, we have deduced that the  $p$ -torsion subgroup of  $A(K_p)$  is infinite.

Of course, this situation is ruled out by the theorem of Imai and Serre mentioned above. Nevertheless, we will sketch for the reader's convenience an argument which leads to a contradiction. Let  $v$  be a place of  $k$  dividing  $p$ , and let  $D \subset G$  be a decomposition group for  $v$ . By ([SGA 71], IX, Prop. 5.6), the  $D$ -module  $V_p$  is an extension of  $D$ -modules attached to  $p$ -divisible groups over the integer ring of the completion of  $k$  at  $v$ . Because of Tate's theory [7], the semisimplification  $V_p^{ss}$  of the  $D$ -module  $V_p$  has a Hodge-Tate decomposition. (Here we should remark that submodules and quotients of Hodge-Tate modules are again Hodge-Tate.) Since  $W$  is semisimple as a  $D$ -module (because semisimple and *abelian* as a  $G$ -module),  $W$  may be viewed as a submodule of  $V_p^{ss}$ . Therefore,  $W$  is a Hodge-Tate module.

By ([6], III, Appendix), we know that  $\rho$  is a locally algebraic abelian representation of  $G$ . Using this information, plus the fact that  $\rho$  factors through  $\text{Gal}(K_p/k)$ , we find that there is an open subgroup  $G_0$  of  $G$  with the following property: the restriction of  $\rho$  to  $G_0$  is the direct sum of 1-dimensional representations, each described by an integral power  $\chi_p^n$  of the standard cyclotomic character  $\chi_p: G \rightarrow \mathbf{Z}_p^*$ . After replacing  $k$  by a finite extension, we may assume that  $G_0$  is  $G$ . Take a prime  $w$  of  $k$  which is prime to  $p$  and such that  $A$  has good reduction at  $w$ . Let  $g \in G$  be a Frobenius element for  $w$ . The eigenvalues of  $\rho(g)$  will be integral powers of  $\chi_p(g)$ , i.e., of the norm  $Nw$  of  $w$ . However, by a well known theorem of Weil, these eigenvalues all have archimedian absolute values equal to  $(Nw)^{1/2}$ . This contradiction completes the proof of Theorem 3.

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(Reçu le 20 janvier 1981)

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