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Artikel: FINITENESS THEOREMS IN GEOMETRIC CLASSFIELD THEORY
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THEOREM 5. *Let S be a normal, connected noetherian scheme, whose function field K is absolutely finitely generated. Let $f: X \to S$ be a smooth surjective morphism of finite type whose geometric generic fibre is connected, and which admits a cross-section $X \overset{\varepsilon}{\leftrightarrows} S$. Then there are only finitely many connected finite etale X-schemes Y/X which are galois over X with abelian galois group of order prime to* char (K) *and which are completely decomposed over the marked section. If in addition we suppose X/S proper, we can drop the proviso "of order prime to* char (K)".

Proof. This is just the concatenation of Theorems 1 and 2 with the physical interpretation (1.3) of the group Ker (X/S) in the presence of a section. QED

V. APPLICATION TO l-ADIC REPRESENTATIONS

Let l be a prime number, $\overline{\mathbf{Q}}_l$ an algebraic closure of $\mathbf{Q}_l$. By an l-adic representation ρ of a topological group π, we mean a finite-dimensional continuous representation

$$\rho: \pi \to GL(n, \overline{\mathbf{Q}}_l)$$

whose image lies in $GL(n, E_\lambda)$ for some finite extension E_λ of $\mathbf{Q}_l$.

THEOREM 6. (cf. Grothendieck, *via* [2], 1.3). *Let K be an absolutely finitely generated field, X/K a smooth, geometrically connected K-scheme of finite type, $\bar{x}$ a geometric point of $X \otimes \overline{K}$, x the image geometric point of $\bar{x}$ in X. Let l be a prime number, and ρ an l-adic representation of $\pi_1(X, x)$;*

$$\rho: \pi_1(X, x) \to GL(n, \overline{\mathbf{Q}}_l).$$

Let G be the Zariski closure of the image $\rho(\pi_1(X \otimes \overline{K}, \bar{x}))$ of the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ in $GL(n, \overline{\mathbf{Q}}_l)$ and G^0 its identity component. Suppose that either l is different from the characteristic p of K, or that X/K is proper. Then:

(1) *the radical of G^0 is unipotent, or equivalently:*

(2) *if the restriction of ρ to the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ is completely reducible, then the algebraic group G^0 is semi-simple.*

Proof. By Theorem 1, for $l \neq p$, or by Theorem 2 if $l = p$ and X/K is *proper*, we know that the l-part of Ker (X/K) is finite i.e. (cf. Lemma 1) the image of $\pi_1 (X \otimes \bar{K}, \bar{x})$ in $\pi_1 (X)^{ab}$ is the product of a finite group and a group of order prime to l. Given this fact, the proof proceeds exactly as in (Deligne [2], 1.3).

QED

Remarks. (1) This theorem is the group-theoretic version of Grothendieck's local monodromy theorem (cf. Serre-Tate ([15], Appendix) for a precise statement, as well as the proof) with X/K "replacing" the spectrum of the fraction field E of a henselian discrete valuation ring with residue field K, and with $\pi_1 (X \otimes \bar{K})$ "replacing" the inertia subgroup I of Gal $(\bar{E}/E)$. The "extra" feature of the "local" case is that the quotient of I by a normal pro-p subgroup is abelian. Therefore any l-adic representation ρ of I, with $l \neq p$, becomes *abelian* when restricted to a suitable open subgroup of I, and hence the associated algebraic group G^0 is automatically abelian. In particular, the radical of G^0 is G^0 itself.

(2) If X/K is itself an abelian variety A/K, then $\pi_1 (A \otimes \bar{K}, \bar{x})$ is abelian. Therefore if l is any prime, and ρ any l-adic representation of $\pi_1 (A \otimes \bar{K}, \bar{x})$, the associated algebraic groups G and G^0 will be abelian; hence if ρ is the restriction to $\pi_1 (A \otimes \bar{K}, \bar{x})$ of an l-adic representation of $\pi_1 (A, x)$, then G^0 is unipotent, i.e. the restriction of ρ to an open subgroup of $\pi_1 (A \otimes \bar{K}, \bar{x})$ is *unipotent* (compare Oort [11], 2).

(3) Can one give an example of X/K proper smooth and geometrically connected over an absolutely finitely generated field K of characteristic $p > 0$ whose fundamental group $\pi_1 (X, x)$ admits an n-dimensional p-adic representation with $n \geqslant 2$ (resp. $n \geqslant 3$) for which the associated algebraic group G^0 is $SL(n)$ (resp. $SO(n)$)? Can we find an abelian scheme A over such an X, all of whose fibres have the same p-rank $n \geqslant 2$, for which the associated p-adic representation of $\pi_1 (X, x)$ has $G^0 = SL(n)$? (cf. Oort [11] for the case of p-rank zero).