Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FINITENESS THEOREMS IN GEOMETRIC CLASSFIELD THEORY
Autor: Katz, Nicholas M. / Lang, Serge
Kapitel: IV. Absolute Finiteness theorems
DOI: https://doi.org/10.5169/seals-51753

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 25.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

By Cartier duality, it is equivalent to show that both Hom $(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and $\mathrm{Ext}^1(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p, and this is obvious (resolve the "first" $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. Absolute Finiteness theorems

THEOREM 3. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$. Let X be a smooth $\mathcal{O}$-scheme of finite type whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O})$ *(i.e. for every prime $\mathfrak{p}$ of $\mathcal{O}$, the fibre over $\mathfrak{p}$, $X \underset{\mathcal{O}}{\otimes} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K, is *finite.* QED

THEOREM 4. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ a finite set of primes of $\mathcal{O}$, $N = p_1 \ldots p_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$ the ring of "integers outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$" in K. Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$-scheme of finite type, whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n])$ *(i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, the fibre*

$$X \otimes (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro-N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$-extension of K is finite times pro-N. QED

THEOREM 5. *Let S be a normal, connected noetherian scheme, whose function field K is absolutely finitely generated. Let $f: X \to S$ be a smooth surjective morphism of finite type whose geometric generic fibre is connected, and which admits a cross-section $X \overset{\varepsilon}{\underset{\to}{\curvearrowleft}} S$. Then there are only finitely many connected finite etale X-schemes Y/X which are galois over X with abelian galois group of order prime to* char (K) *and which are completely decomposed over the marked section. If in addition we suppose X/S proper, we can drop the proviso "of order prime to* char (K)*".*

Proof. This is just the concatenation of Theorems 1 and 2 with the physical interpretation (1.3) of the group Ker (X/S) in the presence of a section. QED

V. APPLICATION TO l-ADIC REPRESENTATIONS

Let l be a prime number, $\overline{\mathbf{Q}}_l$ an algebraic closure of $\mathbf{Q}_l$. By an l-adic representation ρ of a topological group π, we mean a finite-dimensional continuous representation

$$\rho: \pi \to GL(n, \overline{\mathbf{Q}}_l)$$

whose image lies in $GL(n, E_\lambda)$ for some finite extension E_λ of $\mathbf{Q}_l$.

THEOREM 6. (cf. Grothendieck, *via* [2], 1.3). *Let K be an absolutely finitely generated field, X/K a smooth, geometrically connected K-scheme of finite type, $\bar{x}$ a geometric point of $X \otimes \overline{K}$, x the image geometric point of $\bar{x}$ in X. Let l be a prime number, and ρ an l-adic representation of $\pi_1(X, x)$;*

$$\rho: \pi_1(X, x) \to GL(n, \overline{\mathbf{Q}}_l).$$

Let G be the Zariski closure of the image $\rho(\pi_1(X \otimes \overline{K}, \bar{x}))$ of the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ in $GL(n, \overline{\mathbf{Q}}_l)$ and G^0 its identity component. Suppose that either l is different from the characteristic p of K, or that X/K is proper. Then:

(1) *the radical of G^0 is unipotent, or equivalently:*

(2) *if the restriction of ρ to the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ is completely reducible, then the algebraic group G^0 is semi-simple.*