Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FINITENESS THEOREMS IN GEOMETRIC CLASSFIELD THEORY

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Kapitel: IV. Absolute Finiteness theorems

DOI: https://doi.org/10.5169/seals-51753

By Cartier duality, it is equivalent to show that both Hom $(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and $\mathrm{Ext}^1(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p, and this is obvious (resolve the "first" $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. Absolute Finiteness theorems

Theorem 3. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of* $\mathbf{Q}$. *Let X be a smooth $\mathcal{O}$-scheme of finite type whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O})$ *(i.e. for every prime $\mathfrak{p}$ of $\mathcal{O}$, the fibre over $\mathfrak{p}$, $X \underset{\mathcal{O}}{\otimes} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K, is *finite.* QED

Theorem 4. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of* $\mathbf{Q}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *a finite set of primes of $\mathcal{O}$, $N = p_1 \ldots p_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$ the ring of "integers outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$" in K. Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]$-scheme of finite type, whose geometric generic fibre $X \underset{\mathcal{O}}{\otimes} \overline{K}$ is connected, and which maps surjectively to* Spec $(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n])$ *(i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, the fibre*

$$X \otimes (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro-N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1(\mathrm{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \ldots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$-extension of K is finite times pro-N. QED

THEOREM 5. *Let S be a normal, connected noetherian scheme, whose function field K is absolutely finitely generated. Let $f\colon X \to S$ be a smooth surjective morphism of finite type whose geometric generic fibre is connected, and which admits a cross-section $X \overset{\varepsilon}{\leftrightarrows} S$. Then there are only finitely many connected finite etale X-schemes Y/X which are galois over X with abelian galois group of order prime to* char (K) *and which are completely decomposed over the marked section. If in addition we suppose X/S proper, we can drop the proviso "of order prime to* char (K)".

Proof. This is just the concatenation of Theorems 1 and 2 with the physical interpretation (1.3) of the group Ker (X/S) in the presence of a section. QED

V. APPLICATION TO l-ADIC REPRESENTATIONS

Let l be a prime number, $\overline{\mathbf{Q}}_l$ an algebraic closure of $\mathbf{Q}_l$. By an l-adic representation ρ of a topological group π, we mean a finite-dimensional continuous representation

$$\rho : \pi \to GL(n, \overline{\mathbf{Q}}_l)$$

whose image lies in $GL(n, E_\lambda)$ for some finite extension E_λ of $\mathbf{Q}_l$.

THEOREM 6. (cf. Grothendieck, *via* [2], 1.3). *Let K be an absolutely finitely generated field, X/K a smooth, geometrically connected K-scheme of finite type, $\bar{x}$ a geometric point of $X \otimes \overline{K}$, x the image geometric point of $\bar{x}$ in X. Let l be a prime number, and ρ an l-adic representation of $\pi_1(X, x)$;*

$$\rho : \pi_1(X, x) \to GL(n, \overline{\mathbf{Q}}_l)\,.$$

Let G be the Zariski closure of the image $\rho(\pi_1(X \otimes \overline{K}, \bar{x}))$ of the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ in $GL(n, \overline{\mathbf{Q}}_l)$ and G^0 its identity component. Suppose that either l is different from the characteristic p of K, or that X/K is proper. Then:

(1) *the radical of G^0 is unipotent, or equivalently:*

(2) *if the restriction of ρ to the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ is completely reducible, then the algebraic group G^0 is semi-simple.*