III. A VARIANT

Let us agree to call a scheme S *accessible* if there exists an absolutely finitely generated field K for which the set $S(K)$ of K-valued points of S is non-empty. Thus for example, if K is an absolutely finitely generated field, then for *any* subring $R \subset K$, Spec (R) is accessible (by the K-valued point $R \hookrightarrow K$); also any subring R' of the power-series ring $K\left[[X_1, ..., ...]\right]$ over K in any number of variables has Spec (R') accessible

$$(\text{by } R' \hookrightarrow K\,[[X_1, ...]] \overset{X \to 0}{\to} K)\,.$$

On the other hand, the spectrum of a field F is accessible if and only if F is absolutely finitely generated.

THEOREM 2. *Let S be a connected, locally noetherian scheme which is accessible. Let X/S be a proper and smooth S-scheme with geometrically connected fibres. Then the group* Ker (X/S) *is finite.*

Proof. We begin by reducing to the case when S is a finitely generated field. In view of the accessibility of S, this reduction results from the following simple lemma applied with $T = $ Spec (K).

LEMMA 4. *Let X/S be proper and smooth with geometrically connected fibres over a connected locally noetherian scheme S. Given a connected locally noetherian S-scheme T, denote by X_T/T the inverse image of X/S on T, i.e. form the cartesian diagram*

The natural map (cf. 1.5)

$$\text{Ker}\,(X_T/T) \to \text{Ker}\,(X/S)$$

is surjective.

Proof.   Let t be a geometric point of T, s the image geometric point of S, and x a geometric point on the fibre X_s. The homotopy exact sequences (SGA I, Exp X, 1.4) for X/S and X_T/T sit in a commutative diagram

$$
\begin{array}{ccccccc}
\pi_1\,(X_s, x) & \longrightarrow & \pi_1\,(X_T, x) & \longrightarrow & \pi_1\,(T, t) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1\,(X_s, x) & \longrightarrow & \pi_1\,(X, x) & \longrightarrow & \pi_1\,(S, s) & \longrightarrow & 0
\end{array}
$$

Passing to the abelianizations yields the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
\pi_1\,(X_s)^{ab} & \longrightarrow & \pi_1\,(X_T)^{ab} & \longrightarrow & \pi_1\,(T)^{ab} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1\,(X_s)^{ab} & \longrightarrow & \pi_1\,(X)^{ab} & \longrightarrow & \pi_1\,(S)^{ab} & \longrightarrow & 0
\end{array}
$$

whence we find

$$\text{Ker}\,(X_T/T) = \text{image of } \pi_1\,(X_s)^{ab} \text{ in } \pi_1\,(X_T)^{ab}.$$

$$\pi_1\,(X_s)^{ab} \qquad\qquad \downarrow$$

$$\text{Ker}\,(X/S) \quad = \text{image of } \pi_1\,(X_s)^{ab} \text{ in } \pi_1\,(X)^{ab}. \qquad \text{QED}$$

Thus we are reduced to proving the finiteness of $\text{Ker}\,(X/K)$ when K is an absolutely finitely generated field, and X/K is proper, smooth, and geometrically connected. We have already proven this finiteness theorem when X/K is an abelian variety (cf. Remark (1) above). We will reduce to this case by making use of the theory of the Picard and Albanese varieties.

At the expense of replacing K by a finite extension, we may assume that X has a K-rational point x_0. The Picard scheme $Pic_{X/K}$ is then a commutative group-scheme locally of finite type over K, which represents the functor on $\{\text{Schemes}/K\}$

$$
W \to \left\{
\begin{array}{l}
\text{the group of } W\text{-isomorphism classes of pairs } (\mathscr{L}, \varepsilon) \text{ consisting} \\
\text{of an invertible sheaf } \mathscr{L} \text{ on } X \underset{K}{\times} W \text{ together with a} \\
\text{trivialization } \varepsilon \text{ of the restriction } \mathscr{L} \text{ to } \{x_0\} \underset{K}{\times} W
\end{array}
\right.
$$

The subgroup-scheme $Pic^\tau_{X/K}$ of $Pic_{X/K}$ classifies those $(\mathscr{L}, \varepsilon)$ whose underlying $\mathscr{L}$ becomes τ-equivariant to zero when restricted to every geometric fibre of $X \times W/W$ (i.e. for each geometric point w of W, some multiple of $\mathscr{L} \mid X \times w$ is algebraically equivalent to zero). The identity component $Pic^0_{X/K}$ of $Pic_{X/K}$ classifies those $(\mathscr{L}, \varepsilon)$ whose $\mathscr{L}$ becomes algebraically equivalent to zero on each geometric fibre $X \times W/W$. The Picard *variety* $Pic^{0,\,red}_{X/K}$ is an abelian variety over K, and it sits in an *f.p.p.f.* short exact sequence of commutative group schemes

$$(3.1) \qquad 0 \to Pic^{0,\,red}_{X/K} \to Pic^\tau_{X/K} \to C \to 0$$

in which the cokernel C is a finite flat group-scheme over K. This cokernel C should be thought of as the "scheme theoretic" torsion in the Neron-Severi group.

We denote by $\mathrm{Alb}_{X/K}$ the Albanese variety of X/K, *defined* to be the dual abelian variety to the Picard variety $Pic^{0,\,red}_{X/K}$. We now recall the expression of $\pi_1 (X \otimes \overline{K})^{ab}$ in terms of the Tate module of the Albanese, and a finite "error term" involving the Cartier dual $C^\vee$ of C.

LEMMA 5. *Let K be a field, and X/K a proper, smooth and geometrically connected K-scheme which admits a K-rational point. Then there is a canonical short exact sequence of* $\mathrm{Gal}\,(\overline{K}/K)$-*modules*

$$(3.2) \qquad 0 \to C^\vee (\overline{K}) \to \pi_1 (X \otimes \overline{K})^{ab} \to T \left(\mathrm{Alb}_{X/K} (\overline{K})\right) \to 0 .$$

Proof. By Kummer and Artin-Schreier theory, we have for each integer $N \geqslant 1$ a canonical isomorphism

$$\mathrm{Hom}\left(\pi_1 (X \otimes \overline{K})^{ab}, \mathbf{Z}/N\mathbf{Z}\right)$$
$$= H^1_{et} (X \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}) \stackrel{\sim}{\to} \mathrm{Hom}\left(\mu_N, (Pic^\tau_{X/K}) \otimes \overline{K}\right) .$$

in which the last Hom is in the sense of $\overline{K}$-group-schemes. Applying the functor $X \mapsto \mathrm{Hom}\,(\mu_N, X)$ to the short exact sequence

$$0 \to Pic^{0,\,red} \to Pic^\tau \to C \to 0$$

gives a short exact sequence

$$(3.3) \qquad 0 \to \mathrm{Hom}\left(\mu_N, (Pic^{0,\,red}) \otimes \overline{K}\right)$$
$$\to \mathrm{Hom}\left(\mu_N, (Pic^\tau) \otimes \overline{K}\right) \to \mathrm{Hom}\left(\mu_N, C \otimes \overline{K}\right) \to 0$$

(the final zero because over an algebraically closed field, the group $\mathrm{Ext}^1(\mu_N, A)$ vanishes for any abelian variety A, cf. the remark at the end of this section). We now "decode" its two end terms, using Cartier-Nishi duality for the first, and Cartier duality for the last.

The first is

$$\mathrm{Hom}\left(\mu_N, (Pic^{0,\,red}) \otimes \overline{K}\right) = \mathrm{Hom}\left(\mu_N, (Pic^{0,\,red})_N \otimes \overline{K}\right)$$

$$\wr\downarrow \qquad \text{Cartier-Nishi duality}$$

$$\mathrm{Hom}\left((\mathrm{Alb}_{X/N})_N \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}\right)$$

$$\wr\downarrow \qquad \text{evaluation on } \overline{K}\text{-points}$$

$$\mathrm{Hom}\left((\mathrm{Alb}_{X/K}(\overline{K}))_N, \mathbf{Z}/N\mathbf{Z}\right)$$

$$\wr\downarrow$$

$$\mathrm{Hom}\left(T(\mathrm{Alb}_{X/K}(\overline{K})), \mathbf{Z}/N\mathbf{Z}\right).$$

The last is

$$\mathrm{Hom}(\mu_N, C \otimes \overline{K}) \quad \overbrace{\underset{\text{Cartier duality}}{\qquad\qquad}} \quad \mathrm{Hom}(C^{\vee} \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z})$$

$$\Big\downarrow\Big\langle \quad |\text{evaluation}$$

$$\mathrm{Hom}(C^{\vee}(\overline{K}), \mathbf{Z}/N\mathbf{Z})$$

"Substituting" into the exact sequence (3.2), we find a canonical short exact sequence

$$(3.4) \qquad 0 \to \mathrm{Hom}\left(T(\mathrm{Alb}_{X/K}(\overline{K})), \mathbf{Z}/N\mathbf{Z}\right)$$

$$\to \mathrm{Hom}\left(\pi_1(X \otimes \overline{K})^{ab}, \mathbf{Z}/N\mathbf{Z}\right) \to \mathrm{Hom}\left(C^{\vee}(\overline{K}), \mathbf{Z}/N\mathbf{Z}\right) \to 0$$

Passing to the *direct* limit as N grows multiplicatively, we obtain a canonical short exact sequence

$$(3.5) \qquad 0 \to \mathrm{Hom}\left(T(\mathrm{Alb}_X(\overline{K}), \mathbf{Q}/\mathbf{Z}\right)$$

$$\to \mathrm{Hom}\left(\pi_1(X \otimes \overline{K})^{ab}, \mathbf{Q}/\mathbf{Z}\right) \to \mathrm{Hom}\left(C^{\vee}(\overline{K}), \mathbf{Q}/\mathbf{Z}\right) \to 0.$$

Taking its Pontryagin dual, we find the required exact sequence (3.2).   QED

To complete the reduction of Theorem 2 to the case of abelian varieties, we simply notice that the exact sequence of lemma 5 yields, upon passage to coinvariants, an exact sequence

(3.6) $$(C^\vee (\overline{K}))_{\mathrm{Gal}\,/\overline{K}/K)} \to \mathrm{Ker}\,(X/K) \to \mathrm{Ker}\,(\mathrm{Alb}_{X/K}/K) \to 0$$

whose first term, being a quotient of the finite group $C^\vee (\overline{K})$, is finite. QED

Remark. In the course of the proof of Lemma 5, we appealed to the "well-known" vanishing of $\mathrm{Ext}^1 (\mu_N, A)$ over an algebraically closed field, for an abelian variety A and any integer $N > 1$. Here is a simple proof. It is enough to prove this vanishing when N is either prime to the characteristic p of K, or, in case $p > 0$, when $N = p$.

Suppose first N prime to p. Because the ground-field is algebraically closed, we have $\mu_N \simeq \mathbf{Z}/N\mathbf{Z}$, so it is equivalent to prove the vanishing of $\mathrm{Ext}^1 (\mathbf{Z}/N\mathbf{Z}, A)$. We will prove that *this* group vanishes for every integer $N > 1$. Consider such an extension:

$$0 \to A \to E \to \mathbf{Z}/N\mathbf{Z} \to 0$$

Pass to $\overline{K}$-valued points

$$0 \to A\,(\overline{K}) \to E\,(\overline{K}) \to \mathbf{Z}/N\mathbf{Z} \to 0$$

and consider the endomorphism "multiplication by N". Because the group $A\,(\overline{K})$ is N-divisible, the snake lemma gives an exact sequence

$$0 \to A\,(\overline{K})_N \to E\,(\overline{K})_N \to \mathbf{Z}/N\mathbf{Z} \to 0$$

But a point in $E\,(\overline{K})_N$ which maps onto "1" $\in \mathbf{Z}/N\mathbf{Z}$ is precisely a splitting of our extension.

Next consider the case $N = p = \mathrm{char}\,(K)$. We give a proof due to Barry Mazur. Using the *f.p.p.f.* exact sequence

$$0 \to A_p \to A \to A \to 0\,.$$

to compute $\mathrm{Ext}\,(\mu_p, -)$, we obtain a short exact sequence

$$0 \to \mathrm{Hom}\,(\mu_p, A) \to \mathrm{Ext}^1 (\mu_p, A_p) \to \mathrm{Ext}^1 (\mu_p, A) \to 0$$

To prove that $\mathrm{Ext}^1 (\mu_p, A) = 0$, we will show that the groups $\mathrm{Hom}\,(\mu_p, A)$ and $\mathrm{Ext}^1 (\mu_p, A_p)$ are both finite, of the same order. Trivially, we have $\mathrm{Hom}\,(\mu_p, A) = \mathrm{Hom}\,(\mu_p, A_p)$. Because we are over an algebraically closed field, and A_p is killed by p, its toroidal biconnected-etale decomposition looks like

$$A_p \simeq (\mu_p)^a \times (\text{biconnected}) \times (\mathbf{Z}/p\mathbf{Z})^b\,; \qquad [\text{in fact } a = b]\,.$$

Only the $\mu\,p$'s in A_p can "interact" with μ_p. Thus we are reduced to showing that $\mathrm{Hom}\,(\mu_p, (\mu_p)^a)$ and $\mathrm{Ext}^1 (\mu_p, (\mu_p)^a)$ are both finite of the same cardinality p^a.

By Cartier duality, it is equivalent to show that both Hom $(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and Ext^1 $(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p, and this is obvious (resolve the "first" $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0) \, .$$

For another proof in this case, cf. Oort, [10], 85.

IV. Absolute Finiteness theorems

THEOREM 3. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$. Let X be a smooth $\mathcal{O}$-scheme of finite type whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\mathrm{Spec}\,(\mathcal{O})$ (i.e. for every prime $\mathfrak{p}$ of $\mathcal{O}$, the fibre over $\mathfrak{p}$, $X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1\,(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1\,(\mathrm{Spec}\,(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K, is *finite.*   QED

THEOREM 4. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}$, $\mathfrak{p}_1, ..., \mathfrak{p}_n$ a finite set of primes of $\mathcal{O}$, $N = p_1 ... p_n$ the product of their residue characteristics, and $\mathcal{O}\,[1/\mathfrak{p}_1 ... \mathfrak{p}_n]$ the ring of "integers outside $\mathfrak{p}_1, ..., \mathfrak{p}_n$" in K. Let X be a smooth $\mathcal{O}\,[1/\mathfrak{p}_1 ... \mathfrak{p}_n]$-scheme of finite type, whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\mathrm{Spec}\,(\mathcal{O}\,[1/\mathfrak{p}_1 ... \mathfrak{p}_n])$ (i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$, the fibre*

$$X \otimes (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1\,(X)^{ab}$ is the product of a finite group and a pro-N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1\,(\mathrm{Spec}\,(\mathcal{O}\,[1/\mathfrak{p}_1 ... \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$-extension of K is finite times pro-N.   QED