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III. A VARIANT

Let us agree to call a scheme S accessible if there exists an absolutely finitely generated field K for which the set S(K) of K-valued points of S is non-empty. Thus for example, if K is an absolutely finitely generated field, then for any subring $R \subset K$, Spec (R) is accessible (by the K-valued point $R \hookrightarrow K$); also any subring R' of the power-series ring $K[[X_1, ..., ...]]$ over K in any number of variables has Spec (R') accessible

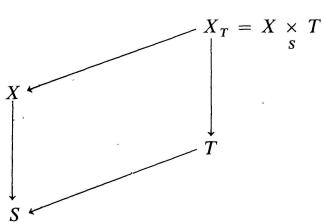
(by
$$R' \subset K [[X_1, ...]]^{X \to 0} \to K$$
).

On the other hand, the spectrum of a field F is accessible if and only if F is absolutely finitely generated.

THEOREM 2. Let S be a connected, locally noetherian scheme which is accessible. Let X/S be a proper and smooth S-scheme with geometrically connected fibres. Then the group Ker(X/S) is finite.

Proof. We begin by reducing to the case when S is a finitely generated field. In view of the accessibility of S, this reduction results from the following simple lemma applied with $T = \operatorname{Spec}(K)$.

Lemma 4. Let X/S be proper and smooth with geometrically connected fibres over a connected locally noetherian scheme S. Given a connected locally noetherian S-scheme T, denote by X_T/T the inverse image of X/S on T, i.e. form the cartesian diagram



The natural map (cf. 1.5)

$$Ker(X_T/T) \rightarrow Ker(X/S)$$

is surjective.

Proof. Let t be a geometric point of T, s the image geometric point of S, and x a geometric point on the fibre X_s . The homotopy exact sequences (SGA I, Exp X, 1.4) for X/S and X_T/T sit in a commutative diagram

Passing to the abelianizations yields the commutative diagram with exact rows

whence we find

$$\pi_1(X_s)^{ab} \xrightarrow{\text{Ker } (X_T/T) = \text{ image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X_T)^{ab} .}$$

$$\pi_1(X_s)^{ab} \xrightarrow{\text{Ker } (X/S) = \text{ image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X)^{ab} .}$$

$$\text{QED}$$

Thus we are reduced to proving the finiteness of Ker(X/K) when K is an absolutely finitely generated field, and X/K is proper, smooth, and geometrically connected. We have already proven this finiteness theorem when X/K is an abelian variety (cf. Remark (1) above). We will reduce to this case by making use of the theory of the Picard and Albanese varieties.

At the expense of replacing K by a finite extension, we may assume that X has a K-rational point x_0 . The Picard scheme $Pic_{X/K}$ is then a commutative group-scheme locally of finite type over K, which represents the functor on $\{Schemes/K\}$

the group of W-isomorphism classes of pairs
$$(\mathcal{L}, \varepsilon)$$
 consisting of an invertible sheaf \mathcal{L} on $X \times W$ together with a trivialization ε of the restriction \mathcal{L} to $\{x_0\} \times W$

The subgroup-scheme $Pic_{X/K}^{\tau}$ of $Pic_{X/K}$ classifies those $(\mathcal{L}, \varepsilon)$ whose underlying \mathcal{L} becomes τ -equivariant to zero when restricted to every geometric fibre of $X \times W/W$ (i.e. for each geometric point w of W, some multiple of $\mathcal{L} \mid X \times w$ is algebraically equivalent to zero). The identity component $Pic_{X/K}^0$ of $Pic_{X/K}$ classifies those $(\mathcal{L}, \varepsilon)$ whose \mathcal{L} becomes algebraically equivalent to zero on each geometric fibre $X \times W/W$. The Picard $variety Pic_{X/K}^{0, red}$ is an abelian variety over K, and it sits in an f.p.p.f. short exact sequence of commutative group schemes

$$(3.1) 0 \to Pic_{X/K}^{0, \text{ red}} \to Pic_{X/K}^{\tau} \to C \to 0$$

in which the cokernel C is a finite flat group-scheme over K. This cokernel C should be thought of as the "scheme theoretic" torsion in the Neron-Severi group.

We denote by $\mathrm{Alb}_{X/K}$ the Albanese variety of X/K, defined to be the dual abelian variety to the Picard variety $Pic_{X/K}^{0,\,\mathrm{red}}$. We now recall the expression of π_1 $(X \otimes \overline{K})^{ab}$ in terms of the Tate module of the Albanese, and a finite "error term" involving the Cartier dual C^{\vee} of C.

LEMMA 5. Let K be a field, and X/K a proper, smooth and geometrically connected K-scheme which admits a K-rational point. Then there is a canonical short exact sequence of $\operatorname{Gal}(\overline{K}/K)$ -modules

$$(3.2) 0 \to C^{\vee}(\overline{K}) \to \pi_1(X \otimes \overline{K})^{ab} \to T(\mathrm{Alb}_{X/K}(\overline{K})) \to 0.$$

Proof. By Kummer and Artin-Schreier theory, we have for each integer $N \ge 1$ a canonical isomorphism

$$\operatorname{Hom}\left(\pi_1\left(X\otimes \overline{K}\right)^{ab}, \mathbf{Z}/N\mathbf{Z}\right) \\ = H^1_{et}\left(X\otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}\right) \cong \operatorname{Hom}\left(\mu_N, \left(\operatorname{Pic}_{X/K}^{\tau}\right) \otimes \overline{K}\right).$$

in which the last Hom is in the sense of \overline{K} -group-schemes. Applying the functor $X \mapsto \operatorname{Hom}(\mu_N, X)$ to the short exact sequence

$$0 \to Pic^{0, \text{ red}} \to Pic^{\tau} \to C \to 0$$

gives a short exact sequence

(3.3)
$$0 \to \operatorname{Hom}\left(\mu_{N}, (Pic^{0, \operatorname{red}}) \otimes \overline{K}\right)$$
$$\to \operatorname{Hom}\left(\mu_{N}, (Pic^{\tau}) \otimes \overline{K}\right) \to \operatorname{Hom}\left(\mu_{N}, C \otimes \overline{K}\right) \to 0$$

(the final zero because over an algebraically closed field, the group $\operatorname{Ext}^1(\mu_N, A)$ vanishes for any abelian variety A, cf. the remark at the end of this section). We now "decode" its two end terms, using Cartier-Nishi duality for the first, and Cartier duality for the last.

The first is

The last is

Hom
$$(\mu_N, C \otimes \overline{K})$$
 $\overbrace{\text{Cartier duality}}$ Hom $(C^{\vee} \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z})$ \downarrow levaluation Hom $(C^{\vee}(\overline{K}), \mathbf{Z}/N\mathbf{Z})$

"Substituting" into the exact sequence (3.2), we find a canonical short exact sequence

(3.4)
$$0 \to \operatorname{Hom}\left(T\left(\operatorname{Alb}_{X/K}(\overline{K})\right), \mathbf{Z}/N\mathbf{Z}\right)$$
$$\to \operatorname{Hom}\left(\pi_1\left(X \otimes \overline{K}\right)^{ab}, \mathbf{Z}/N\mathbf{Z}\right) \to \operatorname{Hom}\left(C^{\vee}(\overline{K}), \mathbf{Z}/N\mathbf{Z}\right) \to 0$$

Passing to the direct limit as N grows multiplicatively, we obtain a canonical short exact sequence

(3.5)
$$0 \to \operatorname{Hom}\left(T\left(\operatorname{Alb}_{X}(\overline{K}), \mathbf{Q}/\mathbf{Z}\right)\right.$$
$$\to \operatorname{Hom}\left(\pi_{1}\left(X \otimes \overline{K}\right)^{ab}, \mathbf{Q}/\mathbf{Z}\right) \to \operatorname{Hom}\left(C^{\vee}\left(\overline{K}\right), \mathbf{Q}/\mathbf{Z}\right) \to 0.$$

Taking its Pontryagin dual, we find the required exact sequence (3.2). QED

To complete the reduction of Theorem 2 to the case of abelian varieties, we simply notice that the exact sequence of lemma 5 yields, upon passage to coinvariants, an exact sequence

$$(3.6) (C^{\vee}(\overline{K}))_{Gal/\overline{K}/K} \to Ker(X/K) \to Ker(Alb_{X/K}/K) \to 0$$

whose first term, being a quotient of the finite group $C^{\vee}(\overline{K})$, is finite. QED

Remark. In the course of the proof of Lemma 5, we appealed to the "well-known" vanishing of $\operatorname{Ext}^1(\mu_N, A)$ over an algebraically closed field, for an abelian variety A and any integer N > 1. Here is a simple proof. It is enough to prove this vanishing when N is either prime to the characteristic p of K, or, in case p > 0, when N = p.

Suppose first N prime to p. Because the ground-field is algebraically closed, we have $\mu_N \simeq \mathbf{Z}/N\mathbf{Z}$, so it is equivalent to prove the vanishing of $\operatorname{Ext}^1(\mathbf{Z}/N\mathbf{Z}, A)$. We will prove that this group vanishes for every integer N > 1. Consider such an extension:

$$0 \rightarrow A \rightarrow E \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

Pass to \overline{K} -valued points

$$0 \to A(\overline{K}) \to E(\overline{K}) \to \mathbf{Z}/N\mathbf{Z} \to 0$$

and consider the endomorphism "multiplication by N". Because the group $A(\overline{K})$ is N-divisible, the snake lemma gives an exact sequence

$$0 \to A(\overline{K})_N \to E(\overline{K})_N \to \mathbf{Z}/N\mathbf{Z} \to 0$$

But a point in $E(\overline{K})_N$ which maps onto "1" $\in \mathbb{Z}/N\mathbb{Z}$ is precisely a splitting of our extension.

Next consider the case N = p = char(K). We give a proof due to Barry Mazur. Using the f.p.p.f. exact sequence

$$0 \to A_n \to A \to A \to 0$$
.

to compute Ext $(\mu_p, -)$, we obtain a short exact sequence

$$0 \to \operatorname{Hom}(\mu_p, A) \to \operatorname{Ext}^1(\mu_p, A_p) \to \operatorname{Ext}^1(\mu_p, A) \to 0$$

To prove that $\operatorname{Ext}^1(\mu_p, A) = 0$, we will show that the groups $\operatorname{Hom}(\mu_p, A)$ and $\operatorname{Ext}^1(\mu_p, A_p)$ are both finite, of the same order. Trivially, we have $\operatorname{Hom}(\mu_p, A) = \operatorname{Hom}(\mu_p, A_p)$. Because we are over an algebraically closed field, and A_p is killed by p, its toroidal biconnected-etale decomposition looks like

$$A_p \simeq (\mu_p)^a \times (\text{biconnected}) \times (\mathbf{Z}/p\mathbf{Z})^b;$$
 [in fact $a = b$].

Only the μ p's in A_p can "interact" with μ_p . Thus we are reduced to showing that Hom $(\mu_p, (\mu_p)^a)$ and Ext¹ $(\mu_p, (\mu_p)^a)$ are both finite of the same cardinality p^a .

By Cartier duality, it is equivalent to show that both Hom $(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ and $\operatorname{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ have order p, and this is obvious (resolve the "first" $\mathbb{Z}/p\mathbb{Z}$ by

$$0 \to \mathbf{Z} \stackrel{p}{\to} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. ABSOLUTE FINITENESS THEOREMS

Theorem 3. Let \mathcal{O} be the ring of integers in a finite extension K of \mathbf{Q} . Let X be a smooth \mathcal{O} -scheme of finite type whose geometric generic fibre $X \otimes \overline{K}$ is connected, and which maps surjectively to $\operatorname{Spec}(\mathcal{O})$ (i.e. for every prime \mathfrak{p} of \mathcal{O} , the fibre over \mathfrak{p} , $X \otimes (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which π_1 (Spec (\mathcal{O}))^{ab}, the galois group of the maximal unramified abelian extension of K, is *finite*. QED

Theorem 4. Let \mathcal{O} be the ring of integers in a finite extension K of $\mathbf{Q}, \mathfrak{p}_1, ..., \mathfrak{p}_n$ a finite set of primes of $\mathcal{O}, N = \mathfrak{p}_1 ... \mathfrak{p}_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 ... \mathfrak{p}_n]$ the ring of "integers outside $\mathfrak{p}_1, ..., \mathfrak{p}_n$ " in K. Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 ... \mathfrak{p}_n]$ -scheme of finite type, whose geometric generic fibre $X \otimes \overline{K}$ is connected, and which maps surjectively

to Spec $(\mathcal{O}[1/\mathfrak{p}_1 ... \mathfrak{p}_n])$ (i.e. for every prime $\mathfrak{p} \notin {\mathfrak{p}_1, ..., \mathfrak{p}_n}$, the fibre

$$X \otimes (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro-N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which π_1 (Spec $(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ -extension of K is finite times pro-N.