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**Autor:** Katz, Nicholas M. / Lang, Serge  
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## III. A VARIANT

Let us agree to call a scheme  $S$  *accessible* if there exists an absolutely finitely generated field  $K$  for which the set  $S(K)$  of  $K$ -valued points of  $S$  is non-empty. Thus for example, if  $K$  is an absolutely finitely generated field, then for *any* subring  $R \subset K$ ,  $\text{Spec}(R)$  is accessible (by the  $K$ -valued point  $R \hookrightarrow K$ ); also any subring  $R'$  of the power-series ring  $K[[X_1, \dots, \dots]]$  over  $K$  in any number of variables has  $\text{Spec}(R')$  accessible

$$(\text{by } R' \hookrightarrow K \xrightarrow{X \rightarrow 0} K).$$

On the other hand, the spectrum of a field  $F$  is accessible if and only if  $F$  is absolutely finitely generated.

**THEOREM 2.** *Let  $S$  be a connected, locally noetherian scheme which is accessible. Let  $X/S$  be a proper and smooth  $S$ -scheme with geometrically connected fibres. Then the group  $\text{Ker}(X/S)$  is finite.*

*Proof.* We begin by reducing to the case when  $S$  is a finitely generated field. In view of the accessibility of  $S$ , this reduction results from the following simple lemma applied with  $T = \text{Spec}(K)$ .

**LEMMA 4.** *Let  $X/S$  be proper and smooth with geometrically connected fibres over a connected locally noetherian scheme  $S$ . Given a connected locally noetherian  $S$ -scheme  $T$ , denote by  $X_T/T$  the inverse image of  $X/S$  on  $T$ , i.e. form the cartesian diagram*

$$\begin{array}{ccc} & X_T = X \times_S T & \\ \swarrow & \downarrow & \\ X & & T \\ \downarrow & \swarrow & \\ S & & \end{array}$$



The natural map (cf. 1.5)

$$\text{Ker } (X_T/T) \rightarrow \text{Ker } (X/S)$$

is surjective.

*Proof.* Let  $t$  be a geometric point of  $T$ ,  $s$  the image geometric point of  $S$ , and  $x$  a geometric point on the fibre  $X_s$ . The homotopy exact sequences (SGA I, Exp X, 1.4) for  $X/S$  and  $X_T/T$  sit in a commutative diagram

$$\begin{array}{ccccccc} \pi_1(X_s, x) & \longrightarrow & \pi_1(X_T, x) & \longrightarrow & \pi_1(T, t) & \longrightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1(X_s, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \pi_1(S, s) & \longrightarrow & 0 \end{array}$$

Passing to the abelianizations yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} \pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X_T)^{ab} & \longrightarrow & \pi_1(T)^{ab} & \longrightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X)^{ab} & \longrightarrow & \pi_1(S)^{ab} & \longrightarrow & 0 \end{array}$$

whence we find

$$\begin{array}{l} \pi_1(X_s)^{ab} \begin{cases} \nearrow \text{Ker } (X_T/T) = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X_T)^{ab} . \\ \searrow \text{Ker } (X/S) = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X)^{ab} . \end{cases} \end{array} \quad \text{QED}$$

Thus we are reduced to proving the finiteness of  $\text{Ker } (X/K)$  when  $K$  is an absolutely finitely generated field, and  $X/K$  is proper, smooth, and geometrically connected. We have already proven this finiteness theorem when  $X/K$  is an abelian variety (cf. Remark (1) above). We will reduce to this case by making use of the theory of the Picard and Albanese varieties.

At the expense of replacing  $K$  by a finite extension, we may assume that  $X$  has a  $K$ -rational point  $x_0$ . The Picard scheme  $\text{Pic}_{X/K}$  is then a commutative group-scheme locally of finite type over  $K$ , which represents the functor on  $\{\text{Schemes}/K\}$

$$W \rightarrow \left\{ \begin{array}{l} \text{the group of } W\text{-isomorphism classes of pairs } (\mathcal{L}, \varepsilon) \text{ consisting} \\ \text{of an invertible sheaf } \mathcal{L} \text{ on } X \times_K W \text{ together with a} \\ \text{trivialization } \varepsilon \text{ of the restriction } \mathcal{L} \text{ to } \{x_0\} \times_K W \end{array} \right.$$



The subgroup-scheme  $Pic_{X/K}^\tau$  of  $Pic_{X/K}$  classifies those  $(\mathcal{L}, \varepsilon)$  whose underlying  $\mathcal{L}$  becomes  $\tau$ -equivariant to zero when restricted to every geometric fibre of  $X \times W/W$  (i.e. for each geometric point  $w$  of  $W$ , some multiple of  $\mathcal{L} \mid X \times w$  is algebraically equivalent to zero). The identity component  $Pic_{X/K}^0$  of  $Pic_{X/K}$  classifies those  $(\mathcal{L}, \varepsilon)$  whose  $\mathcal{L}$  becomes algebraically equivalent to zero on each geometric fibre  $X \times W/W$ . The Picard variety  $Pic_{X/K}^{0, \text{red}}$  is an abelian variety over  $K$ , and it sits in an *f.p.f.* short exact sequence of commutative group schemes

$$(3.1) \quad 0 \rightarrow Pic_{X/K}^{0, \text{red}} \rightarrow Pic_{X/K}^\tau \rightarrow C \rightarrow 0$$

in which the cokernel  $C$  is a finite flat group-scheme over  $K$ . This cokernel  $C$  should be thought of as the “scheme theoretic” torsion in the Neron-Severi group.

We denote by  $Alb_{X/K}$  the Albanese variety of  $X/K$ , defined to be the dual abelian variety to the Picard variety  $Pic_{X/K}^{0, \text{red}}$ . We now recall the expression of  $\pi_1(X \otimes \bar{K})^{ab}$  in terms of the Tate module of the Albanese, and a finite “error term” involving the Cartier dual  $C^\vee$  of  $C$ .

**LEMMA 5.** *Let  $K$  be a field, and  $X/K$  a proper, smooth and geometrically connected  $K$ -scheme which admits a  $K$ -rational point. Then there is a canonical short exact sequence of  $\text{Gal}(\bar{K}/K)$ -modules*

$$(3.2) \quad 0 \rightarrow C^\vee(\bar{K}) \rightarrow \pi_1(X \otimes \bar{K})^{ab} \rightarrow T(Alb_{X/K}(\bar{K})) \rightarrow 0.$$

*Proof.* By Kummer and Artin-Schreier theory, we have for each integer  $N \geq 1$  a canonical isomorphism

$$\begin{aligned} & \text{Hom}(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Z}/N\mathbf{Z}) \\ &= H_{et}^1(X \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z}) \simeq \text{Hom}(\mu_N, (Pic_{X/K}^\tau \otimes \bar{K})). \end{aligned}$$

in which the last  $\text{Hom}$  is in the sense of  $\bar{K}$ -group-schemes. Applying the functor  $X \mapsto \text{Hom}(\mu_N, X)$  to the short exact sequence

$$0 \rightarrow Pic^{0, \text{red}} \rightarrow Pic^\tau \rightarrow C \rightarrow 0$$

gives a short exact sequence

$$(3.3) \quad \begin{aligned} 0 &\rightarrow \text{Hom}(\mu_N, (Pic^{0, \text{red}} \otimes \bar{K})) \\ &\rightarrow \text{Hom}(\mu_N, (Pic^\tau \otimes \bar{K})) \rightarrow \text{Hom}(\mu_N, C \otimes \bar{K}) \rightarrow 0 \end{aligned}$$



(the final zero because over an algebraically closed field, the group  $\text{Ext}^1(\mu_N, A)$  vanishes for any abelian variety  $A$ , cf. the remark at the end of this section). We now “decode” its two end terms, using Cartier-Nishi duality for the first, and Cartier duality for the last.

The first is

$$\begin{aligned}
 \text{Hom}(\mu_N, (\text{Pic}^{0, \text{red}}) \otimes \overline{K}) &= \text{Hom}(\mu_N, (\text{Pic}^{0, \text{red}})_N \otimes \overline{K}) \\
 &\quad \Downarrow \text{Cartier-Nishi duality} \\
 &\text{Hom}(\text{Alb}_{X/N})_N \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}) \\
 &\quad \Downarrow \text{evaluation on } \overline{K}\text{-points} \\
 &\text{Hom}((\text{Alb}_{X/K}(\overline{K}))_N, \mathbf{Z}/N\mathbf{Z}) \\
 &\quad \Downarrow \\
 &\text{Hom}(T(\text{Alb}_{X/K}(\overline{K})), \mathbf{Z}/N\mathbf{Z}).
 \end{aligned}$$

The last is

$$\begin{aligned}
 \text{Hom}(\mu_N, C \otimes \overline{K}) &\xrightarrow{\text{Cartier duality}} \text{Hom}(C^\vee \otimes \overline{K}, \mathbf{Z}/N\mathbf{Z}) \\
 &\quad \downarrow \int \text{evaluation} \\
 &\text{Hom}(C^\vee(\overline{K}), \mathbf{Z}/N\mathbf{Z})
 \end{aligned}$$

“Substituting” into the exact sequence (3.2), we find a canonical short exact sequence

$$\begin{aligned}
 (3.4) \quad &0 \rightarrow \text{Hom}(T(\text{Alb}_{X/K}(\overline{K})), \mathbf{Z}/N\mathbf{Z}) \\
 &\rightarrow \text{Hom}(\pi_1(X \otimes \overline{K})^{ab}, \mathbf{Z}/N\mathbf{Z}) \rightarrow \text{Hom}(C^\vee(\overline{K}), \mathbf{Z}/N\mathbf{Z}) \rightarrow 0
 \end{aligned}$$

Passing to the *direct* limit as  $N$  grows multiplicatively, we obtain a canonical short exact sequence

$$\begin{aligned}
 (3.5) \quad &0 \rightarrow \text{Hom}(T(\text{Alb}_X(\overline{K})), \mathbf{Q}/\mathbf{Z}) \\
 &\rightarrow \text{Hom}(\pi_1(X \otimes \overline{K})^{ab}, \mathbf{Q}/\mathbf{Z}) \rightarrow \text{Hom}(C^\vee(\overline{K}), \mathbf{Q}/\mathbf{Z}) \rightarrow 0.
 \end{aligned}$$

Taking its Pontryagin dual, we find the required exact sequence (3.2). QED

To complete the reduction of Theorem 2 to the case of abelian varieties, we simply notice that the exact sequence of lemma 5 yields, upon passage to coinvariants, an exact sequence



$$(3.6) \quad (C^\vee(\bar{K}))_{\text{Gal}(\bar{K}/K)} \rightarrow \text{Ker}(X/K) \rightarrow \text{Ker}(\text{Alb}_{X/K}/K) \rightarrow 0$$

whose first term, being a quotient of the finite group  $C^\vee(\bar{K})$ , is finite. QED

*Remark.* In the course of the proof of Lemma 5, we appealed to the “well-known” vanishing of  $\text{Ext}^1(\mu_N, A)$  over an algebraically closed field, for an abelian variety  $A$  and any integer  $N > 1$ . Here is a simple proof. It is enough to prove this vanishing when  $N$  is either prime to the characteristic  $p$  of  $K$ , or, in case  $p > 0$ , when  $N = p$ .

Suppose first  $N$  prime to  $p$ . Because the ground-field is algebraically closed, we have  $\mu_N \simeq \mathbf{Z}/N\mathbf{Z}$ , so it is equivalent to prove the vanishing of  $\text{Ext}^1(\mathbf{Z}/N\mathbf{Z}, A)$ . We will prove that *this* group vanishes for every integer  $N > 1$ . Consider such an extension:

$$0 \rightarrow A \rightarrow E \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

Pass to  $\bar{K}$ -valued points

$$0 \rightarrow A(\bar{K}) \rightarrow E(\bar{K}) \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

and consider the endomorphism “multiplication by  $N$ ”. Because the group  $A(\bar{K})$  is  $N$ -divisible, the snake lemma gives an exact sequence

$$0 \rightarrow A(\bar{K})_N \rightarrow E(\bar{K})_N \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

But a point in  $E(\bar{K})_N$  which maps onto “1”  $\in \mathbf{Z}/N\mathbf{Z}$  is precisely a splitting of our extension.

Next consider the case  $N = p = \text{char}(K)$ . We give a proof due to Barry Mazur. Using the *f.p.p.f.* exact sequence

$$0 \rightarrow A_p \rightarrow A \rightarrow A \rightarrow 0.$$

to compute  $\text{Ext}(\mu_p, -)$ , we obtain a short exact sequence

$$0 \rightarrow \text{Hom}(\mu_p, A) \rightarrow \text{Ext}^1(\mu_p, A_p) \rightarrow \text{Ext}^1(\mu_p, A) \rightarrow 0$$

To prove that  $\text{Ext}^1(\mu_p, A) = 0$ , we will show that the groups  $\text{Hom}(\mu_p, A)$  and  $\text{Ext}^1(\mu_p, A_p)$  are both finite, of the same order. Trivially, we have  $\text{Hom}(\mu_p, A) = \text{Hom}(\mu_p, A_p)$ . Because we are over an algebraically closed field, and  $A_p$  is killed by  $p$ , its toroidal biconnected-etale decomposition looks like

$$A_p \simeq (\mu_p)^a \times (\text{biconnected}) \times (\mathbf{Z}/p\mathbf{Z})^b; \quad [\text{in fact } a = b].$$

Only the  $\mu_p$ 's in  $A_p$  can “interact” with  $\mu_p$ . Thus we are reduced to showing that  $\text{Hom}(\mu_p, (\mu_p)^a)$  and  $\text{Ext}^1(\mu_p, (\mu_p)^a)$  are both finite of the same cardinality  $p^a$ .



By Cartier duality, it is equivalent to show that both  $\text{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$  and  $\text{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$  have order  $p$ , and this is obvious (resolve the “first”  $\mathbb{Z}/p\mathbb{Z}$  by

$$0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0).$$

For another proof in this case, cf. Oort, [10], 85.

#### IV. ABSOLUTE FINITENESS THEOREMS

**THEOREM 3.** *Let  $\mathcal{O}$  be the ring of integers in a finite extension  $K$  of  $\mathbb{Q}$ . Let  $X$  be a smooth  $\mathcal{O}$ -scheme of finite type whose geometric generic fibre  $X \otimes_{\mathcal{O}} \overline{K}$  is connected, and which maps surjectively to  $\text{Spec}(\mathcal{O})$  (i.e. for every prime  $\mathfrak{p}$  of  $\mathcal{O}$ , the fibre over  $\mathfrak{p}$ ,  $X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$ , is non empty). Then the group  $\pi_1(X)^{ab}$  is finite.*

*Proof.* This follows immediately from Theorem 1 and global classfield theory, according to which  $\pi_1(\text{Spec}(\mathcal{O}))^{ab}$ , the galois group of the maximal unramified abelian extension of  $K$ , is finite. QED

**THEOREM 4.** *Let  $\mathcal{O}$  be the ring of integers in a finite extension  $K$  of  $\mathbb{Q}$ ,  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  a finite set of primes of  $\mathcal{O}$ ,  $N = p_1 \dots p_n$  the product of their residue characteristics, and  $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$  the ring of “integers outside  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ” in  $K$ . Let  $X$  be a smooth  $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$ -scheme of finite type, whose geometric generic fibre  $X \otimes_{\mathcal{O}} \overline{K}$  is connected, and which maps surjectively to  $\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n])$  (i.e. for every prime  $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ , the fibre*

$$X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$$

*is non-empty). Then the group  $\pi_1(X)^{ab}$  is the product of a finite group and a pro- $N$  group.*

*Proof.* Again an immediate consequence of Theorem 1 and global classfield theory, according to which  $\pi_1(\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]))^{ab}$ , the galois group of the maximal abelian, unramified outside  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ -extension of  $K$  is finite times pro- $N$ . QED