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(4) This follows immediately from (3), and the commutativity of the diagram of maps induced by the obvious inclusions

$$\begin{array}{ccc} \pi_1 (X_\eta)^{ab} & \longrightarrow & \pi_1 (X_\eta)^{ab} \\ & \searrow & \swarrow \\ & \pi_1 (X)^{ab} & \end{array}$$

LEMMA 3. Let X be a smooth geometrically connected variety of finite type over a field K , and let $U \subset X$ be any non-empty open set. Then the natural map

$$\text{Ker} (U/K) \rightarrow \text{Ker} (X/K)$$

is surjective.

Proof. The variety $X \otimes \bar{K}$ is normal and connected, as is the non-empty open $U \otimes \bar{K}$ in it. Therefore the natural map $\pi_1 (U \otimes \bar{K}) \rightarrow \pi_1 (X \otimes \bar{K})$ is surjective (because both source and target are quotients of the galois group of their common function field). The result now follows from the indicated surjectivities in the commutative diagram

$$\begin{array}{ccc} \pi_1 (U \otimes \bar{K})^{ab} & \longrightarrow & \text{Ker} (U/K) \\ \downarrow & & \downarrow \\ \pi_1 (X \otimes \bar{K})^{ab} & \longrightarrow & \text{Ker} (X/K) \end{array}$$

II. THE MAIN THEOREM

Recall that a field K is said to be absolutely finitely generated if it is a finitely generated extension of its prime field, i.e. of \mathbf{Q} or of \mathbf{F}_p .

THEOREM 1. Let S be a normal, connected, locally noetherian scheme, whose function field K is an absolutely finitely generated field. Let $f: X \rightarrow S$ be a smooth surjective morphism of finite type, whose geometric generic fibre is connected. Then the group $\text{Ker} (X/S)$ is finite if K has characteristic zero, and it is the product of a finite group with a pro- p group in case K has characteristic p .

Proof. We will first reduce to the case in which X/S is an elementary fibration in the sense of M. Artin (SGA 4, Exp XI, 3.1), i.e. the complement, in a proper and smooth curve C/S with geometrically connected fibres, of a divisor $D \subset C$ which is finite etale over S . By lemma 2, part (4), $\text{Ker}(X/S)$ is a quotient of $\text{Ker}(X_{\eta}/K)$, so we are reduced to the case $S = \text{Spec}(K)$. If L is a finite extension of K , then $\text{Ker}(X/K)$ is a quotient of $\text{Ker}(X \otimes L/L)$ (by lemma 1), so we may further reduce to the case when X/K has a K -rational point, say x_0 . Thanks to M. Artin's theory of good neighborhoods (SGA 4, Exp XI, 3.3), at the expense of once again passing to a finite extension field L of K , we can find a Zariski open neighborhood U of x_0 in $X \otimes_K L$ which sits atop a finite tower

$$\begin{array}{ccc}
 U = U_0 & & \\
 \downarrow & f_0 & \\
 U_1 & & \\
 \downarrow & f_2 & \\
 (2.1) \quad U_2 & & \\
 \downarrow & & \\
 \vdots & & \\
 \downarrow & & \\
 U_n = \text{Spec}(L) & &
 \end{array}$$

in which each morphism f_i is an elementary fibration. By lemma 1 again, it suffices to prove the theorem for $X \otimes L/L$, and for this it suffices, by lemma 3, to prove it for a good neighborhood U/L . By the exact sequence (1.4), it suffices to prove the theorem for each step U_i/U_{i+1} individually.

This completes the reduction to the case of an elementary fibration. By lemma 2, part (4) we may further reduce to the case $S = \text{Spec}(K)$. Again passing to a finite extension L/K , which is allowable by lemma 1, we may assume that our elementary fibration $X/K (= (C - D)/K)$ has a K -rational point x_0 and that the divisor D of points at infinity consists of a finite set of distinct K rational points of C . We must show that the prime-to- p -part ($p = \text{char}(K)$) of the group of Galois coinvariants

$$(\pi_1(X \otimes \bar{K})^{ab})_{\text{Gal}(\bar{K}/K)}$$

is finite.

For this, we must recall the explicit description of the prime-to- p part of $\pi_1(X \otimes \bar{K})^{ab}$ as the Tate module of a generalized Jacobian. Let J denote the Jacobian $\text{Pic}_{C/K}^0$, and let J_D denote the generalized Jacobian of C/K with respect to the modulus D . Thus J_D is a smooth commutative group-scheme over K which represents the functor on $\{\text{schemes}/K\}$

$$(2.2) \quad W \longrightarrow \left\{ \begin{array}{l} \text{the group of } W\text{-isomorphism classes of pairs } (\mathcal{L}, \varepsilon) \text{ consisting} \\ \text{of an invertible sheaf } \mathcal{L} \text{ on } C \times_K W \text{ which is fibre-by-fibre of} \\ \text{degree zero, together with a trivialization } \varepsilon \text{ of the restriction} \\ \text{of } \mathcal{L} \text{ to } D \times W. \end{array} \right.$$

“Forgetting ε ” defines a natural map $J_D \rightarrow J$, which makes J_D an extension of J by a $\#(D) - 1$ dimensional split torus:

$$(2.3) \quad 0 \rightarrow (\mathbf{G}_m)^{\#(D)}/\mathbf{G}_m \rightarrow J_D \rightarrow J \rightarrow 0.$$

Kummer theory (cf. SGA 4, Exp. XVIII, 1.6 for a “modern” account) furnishes a canonical isomorphism between the prime-to- p part of $\pi_1(X \otimes \bar{K})^{ab}$ and the prime-to- p Tate module of J_D ; for any finite abelian group G killed by an integer N prime to the characteristic p of K , it gives a canonical isomorphism

$$(2.4) \quad H_{\text{et}}^1(X \otimes \bar{K}, G) \simeq \text{Hom}(J_D(\bar{K}))_N, G$$

where $(J_D(\bar{K}))_N$ is the “abstract” subgroup of points of order N in $J_D(\bar{K})$. In terms of the prime-to- p Tate module

$$(2.5) \quad T_{\text{not } p}(J_D(\bar{K})) \stackrel{\text{def}}{=} \varprojlim_{p \nmid N} (J_D(\bar{K}))_N \\ \simeq \prod_{l \neq p} T_l(J_D(\bar{K})),$$

we can rewrite this

$$(2.6) \quad \text{Hom}(\pi_1(X \otimes \bar{K})^{ab}, G) \simeq \text{Hom}(T_{\text{not } p}(J_D(\bar{K})), G),$$

whence finally a canonical isomorphism

$$(2.7) \quad \pi_1(X \otimes \bar{K})^{ab} \simeq T_{\text{not } p}(J_D(\bar{K})) \times (\text{a pro-}p\text{-group}).$$

Thus we are reduced to showing the finiteness of the group

$$(T_{\text{not } p}(J_D(\bar{K})))_{\text{Gal}(\bar{K}/K)}.$$

The exact sequence (2.3)

$$0 \rightarrow (\mathbf{G}_m)^{\#(D)-1} \rightarrow J_D \rightarrow J \rightarrow 0$$

gives an exact sequence of \bar{K} -valued points

$$0 \rightarrow \mathbf{G}_m(\bar{K})^{\#(D)-1} \rightarrow J_D(\bar{K}) \rightarrow J(\bar{K}) \rightarrow 0$$

Applying the snake lemma to the endomorphism “multiplication by N ” of this exact sequence, and passing to the inverse limit over N 's prime to p , we get a short exact sequence of prime-to- p Tate modules

$$(2.8) \quad 0 \rightarrow T_{\text{not } p}(\mathbf{G}_m(\bar{K}))^{\#(D)-1} \rightarrow T_{\text{not } p}(J_D(\bar{K})) \rightarrow T_{\text{not } p}(J(\bar{K})) \rightarrow 0.$$

Because formation of $\text{Gal}(\bar{K}/K)$ -coinvariants is right-exact, we are reduced to showing separately the finiteness of the groups

$$(T_{\text{not } p}(\mathbf{G}_m(\bar{K})))_{\text{Gal}(\bar{K}/K)}, \quad (T_{\text{not } p}(J(\bar{K})))_{\text{Gal}(\bar{K}/K)}.$$

In fact, these groups are finite even if we replace $T_{\text{not } p}$ by the entire Tate module $T = T_p \times T_{\text{not } p}$.

THEOREM 1 (bis). *Let K be an absolutely finitely generated field, and A/K an abelian variety. The groups*

$$(T(\mathbf{G}_m(\bar{K})))_{\text{Gal}(\bar{K}/K)}, \quad T(A(\bar{K}))_{\text{Gal}(\bar{K}/K)}$$

are finite.

Proof. We will reduce to the case when K is finite. Because K is absolutely finitely generated, it is standard that we can find an integrally closed sub-ring R of K , with fraction K , which is finitely generated as a \mathbf{Z} -algebra, together with an abelian scheme \mathbf{A} over R whose generic fibre $\mathbf{A} \otimes_R K$ is A . If K has characteristic $p > 0$, we may further suppose that geometric fibres of \mathbf{A}/R have constant p -rank (if $g = \dim \mathbf{A}/R$, simply localize on R until the rank of the g 'th iterate of the p -linear Hasse-Witt operation on $H^1(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$ is constant).

Suppose first that K has characteristic $p > 0$. Then the $\text{Gal}(\bar{K}/K)$ representations $T(\mathbf{G}_m(\bar{K}))$ and $T(A(\bar{K}))$ are unramified over $\text{Spec}(R)$, i.e. they are actually representations of the fundamental group $\pi_1(\text{Spec}(R), \bar{\eta})$, viewed as a quotient of $\text{Gal}(\bar{K}/K)$.

Let \mathfrak{p} be a maximal ideal of R , i.e. a closed point of $\text{Spec}(R)$, $\mathbb{F}_{\mathfrak{p}}$ its residue field, $\overline{\mathbb{F}}_{\mathfrak{p}}$ an algebraic closure of $\mathbb{F}_{\mathfrak{p}}$, and $\bar{\mathfrak{p}}$ the corresponding geometric point of $\text{Spec}(R)$ (namely $R \rightarrow R/\mathfrak{p} = \mathbb{F}_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_{\mathfrak{p}}$). Pick a “chemin” from \mathfrak{p} to the geometric generic point $\bar{\eta}$ (which is $R \hookrightarrow K \hookrightarrow \bar{K}$), i.e. letting R denote the integral closure of R in \bar{K} , pick a homomorphism $\bar{R} \rightarrow \overline{\mathbb{F}}_{\mathfrak{p}}$ which extends $\bar{\mathfrak{p}}$. Then we get isomorphisms of $\hat{\mathbf{Z}}$ -modules

$$T(\mathbf{A}(\overline{\mathbb{F}}_{\mathfrak{p}})) \xleftarrow[\substack{\sim \\ \text{chosen chemin} \\ \bar{R} \rightarrow \overline{\mathbb{F}}_{\mathfrak{p}}}]{} T(\mathbf{A}(\bar{R})) \xrightarrow[\substack{\sim \\ \bar{R} \hookrightarrow \bar{K}}]{} T(\mathbf{A}(\bar{K}))$$

which is $\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$ equivariant when we make $\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$ operate on $T(\mathbf{A}(\bar{K}))$ via the composite

$$\begin{aligned} \text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) &= \pi_1(\text{Spec}(\mathbb{F}_{\mathfrak{p}}); \bar{\mathfrak{p}}) \xrightarrow{\text{“p”}} \\ \pi_1(\text{Spec}(R), \bar{\mathfrak{p}}) &\xrightarrow[\sim]{\text{“chemin”}} \pi_1(\text{Spec}(R), \bar{\eta}) \end{aligned}$$

Passing to coinvariants now yields a diagram

$$\begin{array}{ccc} (T(\mathbf{A}(\overline{\mathbb{F}}_{\mathfrak{p}})))_{\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})} & \xrightarrow{\cong} & (T(\mathbf{A}(\bar{K})))_{\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})} \\ & & \downarrow \\ & & (T(\mathbf{A}(\bar{K})))_{\pi_1(\text{Spec}(R), \bar{\eta})} \\ & & \parallel \\ & & T(\mathbf{A}(\bar{K}))_{\text{Gal}(\bar{K}/K)}, \end{array}$$

in which the vertical arrow is trivially surjective (because $\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$ operates through its image in $\pi_1(\text{Spec}(R), \bar{\eta})$). Similarly for \mathbf{G}_m .

When K is of characteristic zero, and \mathbf{A}/K has been “spread out” to an abelian scheme \mathbf{A}/R , we argue as follows. Fix a closed point \mathfrak{p} of $\text{Spec}(R)$. For each prime $l \neq p = \text{char}(\mathbb{F}_{\mathfrak{p}})$, the l -adic Tate module $T_l(\mathbf{A}(\bar{K}))$ is unramified over $\text{Spec}(R[1/l])$ and the above specialization argument gives a surjection, for each $l \neq p$,

$$T_l(\mathbf{A}(\overline{\mathbb{F}}_{\mathfrak{p}}))_{\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})} \twoheadrightarrow T_l(\mathbf{A}(\bar{K}))_{\text{Gal}(\bar{K}/K)}.$$

Therefore the prime-to- p part of the order of $(T(A(\bar{K})))_{\text{Gal}(\bar{K}/K)}$ divides the order of $(T(A(\bar{\mathbb{F}}_p)))_{\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)}$.

Now choose a second closed point λ of $\text{Spec}(R)$, with residue characteristic $l \neq p$. [This is possible because, K being a characteristic zero, $\text{Spec}(R)$ necessarily dominates $\text{Spec}(\mathbb{Z})$, and hence by Chevalley's theorem all but finitely many primes occur as residue characteristics of closed points of $\text{Spec}(R)$]. Then the p -part (and indeed the prime-to- l part) of the order of $(T(A(\bar{K})))_{\text{Gal}(\bar{K}/K)}$ divides the order of $(T(A(\bar{\mathbb{F}}_\lambda)))_{\text{Gal}(\bar{\mathbb{F}}_\lambda/\mathbb{F}_\lambda)}$. Similarly for G_m .

Thus we have reduced theorem 1 (bis) to the case of finite fields, where it is "classical". Explicitely, the result is

THEOREM 1 (ter). *Let k be a finite field, $q = \#k$, and A an abelian variety over k . Then we have the explicit formulas*

$$\begin{cases} \#(T(A(\bar{k})))_{\text{Gal}(\bar{k}/k)} = \#A(k) \\ \#(T(G_m(\bar{k})))_{\text{Gal}(\bar{k}/k)} = \#G_m(k) = q - 1. \end{cases}$$

Proof. Let $F \in \text{Gal}(\bar{k}/k)$ denote the arithmetic Frobenius automorphism of \bar{k}/k (i.e. $F(x) = x^q$) which is a topological generator of $\text{Gal}(\bar{k}/k)$. In any $\text{Gal}(\bar{k}/k)$ -module T , the coinvariants are simply the cokernel of $1 - F$:

$$T / (1 - F) T \simeq (T)_{\text{Gal}(\bar{k}/k)}.$$

In the case $T = T(G_m(\bar{k}))$, T is a free module of rank one over $\prod_{l \neq p} \mathbb{Z}_l$ on which F operates as multiplication by q , whence the asserted result. In the case $T = T(A(\bar{k}))$, we have $T = \prod_l T_l(A(\bar{k}))$, the product extended to all primes l .

Each module $T_l(A(\bar{k}))$ is a free \mathbb{Z}_l -module of finite rank ($2 \dim A$ for $l \neq p$, the "p-rank" of A for $l = p$). Because $\#A(k)$ is non-zero, it is enough to prove that, for each l , we have an equality of l -adic ordinals:

$$\text{ord}_l(\#(T_l(A(\bar{k}))/ (1 - F) T_l(A(\bar{k})))) = \text{ord}_l(\#A(k)).$$

By the theory of elementary divisors, we have

$$\text{ord}_l(\#(T_l / (1 - F) T_l)) = \text{ord}_l(\det(1 - F | T_l)).$$

Now for $l \neq p$, we have Weil's celebrated equality ([16], thm. 36)

$$\det(1 - F | T_l(A(\bar{k}))) = \#A(k) \quad (l \neq p).$$

For $l = p$, we have (cf. [13]) only the weaker, but adequate

$$\det (1 - F | T_p (A(\bar{k}))) = (\# A(k)) \times (\text{a } p\text{-adic unit}). \quad \text{QED}$$

Remarks. (1) Given an abelian variety A over any field K , Kummer theory and duality lead to a canonical isomorphism

$$\pi_1 (A \otimes \bar{K}) \simeq T(A(\bar{K})).$$

Because abelian varieties have rational points (e.g. their origins) we have canonically

$$\text{Ker} (A/K) \simeq (T(A(\bar{K})))_{\text{Gal}(\bar{K}/K)}.$$

From this point of view, Theorem 1 (bis) is simply the abelian variety case of Theorem 1 with the added information that even the p -part is finite.

Now consider the special case when $K = k$ is a *finite* field. Then Theorem 1 (ter) gives us

$$\# \text{Ker} (A/k) = \# A(k).$$

In fact, there is a canonical isomorphism of groups

$$\text{Ker} (A/k) \simeq A(k).$$

To see this recall the interpretation of $\text{Ker} (A/k)$ as the inverse limit of the galois groups of connected finite etale A -schemes E/A which are galois over A with abelian galois group, and completely decomposed over the origin (cf. 1.3). The Lang isogeny

$$\begin{array}{ccc} A & & \\ \downarrow 1-F & (F \text{ the Frobenius endomorphism of } A/k) & \\ A & & \end{array}$$

is precisely such a covering, with structural group $A(k)$. Therefore we have a surjective homomorphism

$$\text{Ker} (A/k) \twoheadrightarrow A(k)$$

which is the required isomorphism (since source and target have the same cardinality!).

(2) The \mathbf{G}_m case of Theorem 1 (bis) could have been handled directly by remarking that for any field K , the cardinality (as a supernatural number) of the group of coinvariants $(T(\mathbf{G}_m(\bar{K})))_{\text{Gal}(\bar{K}/K)}$ is equal to the number of roots of unity in the field K . But how, in fact, do we know that this number is finite for an

absolutely finitely generated field? The proof by specialization is pretty much the simplest one! Another approach, after “fattening” K into its finitely generated sub-ring R , is to prove the stronger assertion, in Mordell-Weil style, that the group $G_m(R) = R^\times$ of units in such an absolutely finitely ring is a finitely generated abelian group.

(3) In the case of an abelian variety A over an absolutely finitely generated field K , the multiplicative upper bounds we get for $\# T(A(\overline{K}))_{\text{Gal}(\overline{K}/K)}$ (essentially $\# A(k)$ whenever we specialize to a finite field k , with the proviso that we must ignore the p -parts when it’s a mixed-characteristic specialization) are *exactly the same bounds* usually used to control the size of the torsion subgroup of $A(K)$. There is a simple galois-theoretic interpretation of the group $(T(A(\overline{K})))_{\text{Gal}(\overline{K}/K)}$, or at least its prime-to- p part, in terms of “*twisted-rational*” torsion points, which is perhaps worth pointing out. Thus let A^\vee denote the dual abelian variety to A , p the characteristic of K , $\text{Tors}_{\text{not } p} A^\vee(\overline{K})$ the $\text{Gal}(\overline{K}/K)$ -module of all torsion points of order prime-to- p on A^\vee and

$$(\text{Tors}_{\text{not } p} A^\vee(\overline{K}))(-1)$$

the $\text{Gal}(\overline{K}/K)$ -module obtained from this one by tensoring with the *inverse* of the cyclotomic character χ of $\text{Gal}(\overline{K}/K)$. Alternately, we could describe this last module as the $\text{Gal}(\overline{K}/K)$ -module

$$\text{Hom}(T(G_m(\overline{K})), \text{Tors}_{\text{not } p} A^\vee(\overline{K})).$$

The e_N -pairings define a $\text{Gal}(\overline{K}/K)$ -equivariant pairing

$$T_{\text{not } p}(A(\overline{K})) \times (\text{Tors}_{\text{not } p} A^\vee(\overline{K}))(-1) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

which makes the compact abelian group $T_{\text{not } p}$ and the discrete abelian group $(\text{Tors}_{\text{not } p})(-1)$ the Pontryagin duals of each other. Thus we obtain a perfect pairing

$$T_{\text{not } p}(A(\overline{K}))_{\text{Gal}(K/K)} \times ((\text{Tors}_{\text{not } p}(A^\vee(\overline{K}))(-1))^{\text{Gal}(\overline{K}/K)}) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

The group $((\text{Tors}_{\text{not } p} A^\vee(\overline{K}))(-1))^{\text{Gal}(\overline{K}/K)}$ is none other than the group $(\text{Tors}_{\text{not } p} A^\vee(\overline{K}))^\chi$ of all prime-to- p ($p = \text{char}(K)$) torsion points in $A^\vee(\overline{K})$ which transform under $\text{Gal}(\overline{K}/K)$ by the cyclotomic character χ . Thus we obtain

SCHOLIE. *Over any field K of characteristic zero, the Pontryagin dual of $\text{Ker}(A/K)$ is the group $(\text{Tors } A^\vee(\overline{K}))^\chi$.*

(4) The same reasoning as in (3) above, if carried “scheme-theoretically”, leads to a concrete interpretation of the Pontryagin dual of the entire group $T(A(\overline{K}))_{\text{Gal}(\overline{K}/K)}$ “in terms of” μ -type subgroup schemes” of A^\vee ;

SCHOLIE. *Over any field K , the Pontryagin dual of the compact group $T(A(\overline{K}))_{\text{Gal}(\overline{K}/K)}$ is the discrete group*

$$\varinjlim_N \text{Hom}_{K\text{-gp}}(\mu_N, A^\vee),$$

where Hom is taken in the category of K -group schemes, and the transition maps are those induced by $\mu_{NM} \xrightarrow{\text{“}M\text{”}} \mu_N$.

Still by Theorem 1 (bis), this group is *finite* for an absolutely finitely generated field K .

For any given curve X over, say, \mathbf{Q} , it is an interesting problem to compute the maximal μ -type subgroup of its Jacobian. For example, let p be an odd prime, and consider the modular curves $X_0(p)$ and $X_1(p)$. Then $X_1(p)$ is a ramified covering of $X_0(p)$, cyclic of degree $(p-1)/2$, which is completely split over the rational cusp at infinity. Let

$$N = \text{numerator of } (p-1)/12.$$

The unique intermediate covering of $X_0(p)$ of degree N is unramified; it is called the Shimura covering. According to Mazur [20], the corresponding μ_N inside $J_0(p)$ is the maximal μ -type subgroup of $J_0(p)$ over \mathbf{Q} . Therefore we have

$$\text{Ker}(X_0(p)/\mathbf{Q}) \simeq \mathbf{Z}/N\mathbf{Z}$$

with the Shimura covering as the maximal abelian unramified geometric covering of $X_0(p)$ defined over \mathbf{Q} in which the rational cusp at infinity splits completely.

On the other hand, we may extend $X_0(p)$ to a normal scheme $\mathbf{X}_0(p)$ over \mathbf{Z} . At the prime p , the covering $X_1(p)$ (and hence also the Shimura covering) becomes *completely* ramified over one of the two components of $\mathbf{X}_0(p) \otimes \mathbf{F}_p$. Therefore

$$\text{Ker}(\mathbf{X}_0(p)/\mathbf{Z}) = 0,$$

so that $\text{Spec}(\mathbf{Z})$ being simply connected, we have

$$\pi_1(\mathbf{X}_0(p)^{ab}) = 0.$$

(5) Consider the case when K is a finitely generated extension of an algebraically closed constant field K_0 , and suppose that A/K is an abelian variety over K which has *no fixed part* relative to K_0 . Because K_0 , and hence K , contains all roots of unity, the cyclotomic character of $\text{Gal}(\bar{K}/K)$ is trivial. Therefore the Pontryagin dual of $T_{\text{not } p}(A(\bar{K}))_{\text{Gal}(\bar{K}/K)}$ is simply the group of K -rational torsion points of prime-to- p order on A^\vee . By the *Mordell-Weil theorem* in the function field case (cf. [4], V, thm. 2) the group $A(K)$ of all K -rational points on A is finitely generated so in particular its torsion subgroup is finite. Therefore the group $T_{\text{not } p}(A(\bar{K}))_{\text{Gal}(\bar{K}/K)}$ is also finite in this "geometric" case.

Whether or not the p -part $(T_p(A(\bar{K}))_{\text{Gal}(\bar{K}/K)})$ is also finite under these assumptions is unknown in general. When A/K is a non-constant elliptic curve, this finiteness can be established by considering the ramification properties of the " V -divisible group" of A near a supersingular point on the moduli scheme. However, the general case would seem to require new ideas.

(6) Theorem 1 (bis) implies the finiteness of the group $(\text{Tors } A^\vee(\bar{K}))^\times$ when K is a finitely generated extension of \mathbf{Q} , e.g. a number field. Let $K(\mu)$ be the field obtained by adjoining to K all roots of unity. We clearly have the inclusion

$$(\text{Tors } A^\vee(\bar{K}))^\times \subset \text{Tors } A^\vee(K(\mu)).$$

This leads to the conjecture:

For any abelian variety A over a number field K , the group $\text{Tors } A(K(\mu))$ of $K(\mu)$ -rational torsion points on A is finite.

When A is an elliptic curve without complex multiplication, this is an immediate consequence of Serre's theorem that the Galois group of the torsion points is open in $\prod GL_2(\mathbf{Z}_p)$.

For an arbitrary abelian variety, Imai [Im] shows that the group of torsion points in $K(\mu_{p^\infty})$ is finite for a fixed prime p . We shall prove below that the conjecture is true when A admits complex multiplication. This was extended to a proof of the conjecture in general by Ribet, cf. the appendix.

First we need a lemma.

LEMMA. *Let k be a number field. There exists a positive integer m such that, if F is any finite extension of k ramified at only one prime number p , and contained in some cyclotomic field, then*

$$F \subset k(\mu_{p^\infty}, \mu_m).$$

Proof. There exists a finite set of primes S such that

$$\text{Gal}(k(\mu)/k) = G_S \times \prod_{l \notin S} G_l$$

where $G_l \approx \mathbf{Z}_l^*$, and G_S contains a subgroup

$$H_S = \prod_{l \in S} H_l$$

where H_l is open in \mathbf{Z}_l^* . Without loss of generality, we may assume that S contains p and all primes which ramify in k . If $l \notin S$, then the inertia group at l contains G_l (embedded as a component of the product). If $l \in S$, then the inertia group at l contains a subgroup H'_l open in H_l . Consequently the subgroup of the Galois group generated by all the inertia groups at primes $l \neq p$ contains

$$\prod_{\substack{l \in S \\ l \neq p}} H'_l \times \prod_{l \notin S} G_l.$$

This proves the lemma.

Now let A be an abelian variety defined over a number field k , and with complex multiplication. Suppose that $A_{\text{tor}}(k(\mu))$ is infinite, so contains points of arbitrarily high order. We consider separately the two cases when there is a point of prime order p rational over $k(\mu)$ for arbitrarily large p , or when for some fixed p , there is a point of order p^n with large n .

After extending k by a finite extension if necessary, we may assume without loss of generality that A has good reduction at every prime of k . Let $k' = k(\mu_m)$ where m is chosen as in the lemma. Let x be a point on A of order a power of the prime p . Then $k(x)$ is ramified only at p , and it follows that

$$k'(x) \subset k'(\mu_{p^\infty}).$$

Let K be the field of complex multiplication, which we may also assume contained in k' . Furthermore, after an isogeny of A if necessary, we may assume that the ring of algebraic integers in K acts on A via an embedding

$$\iota: \mathfrak{o}_K \rightarrow \text{End}(A).$$

Let

$$\mathfrak{p}\mathfrak{o}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

be the prime ideal decomposition of p in K , and let $\mathfrak{p}_1 = \mathfrak{p}$, say.

Suppose that x has order p , and that p is large, so p is unramified in k' . By projection on the \mathfrak{p} -component, we may assume that x is a point of order \mathfrak{p} , that is $\iota(\mathfrak{p})x = 0$. If $r \geq 2$, and \mathfrak{P}' is a prime ideal of k' dividing one of $\mathfrak{p}_2, \dots, \mathfrak{p}_r$, then

\mathfrak{P}' is unramified in $k'(x)$. But since p is unramified in k' , then $k'(\mu_{p^\infty})$ is totally ramified above every prime dividing p in k' . Therefore $r = 1$ and p remains prime in k' .

In that case, $k'(x) = k'(A_p)$ and A_p is a cyclic module over \mathfrak{o}_K , or also a vector space of dimension 1 over $\mathfrak{o}_K/p\mathfrak{o}_K$. Furthermore, $\text{Gal}(k'(A_p)/k')$ can be identified with a subgroup of $(\mathfrak{o}_K/p\mathfrak{o}_K)^*$, which has order $Np - 1$, and in particular is prime to p . By a theorem of Ribet [Ri], we have

$$|\text{Gal}(k'(A_p)/k)| \gg p^2,$$

where the sign \gg means that the left hand side is greater than some positive constant times the right hand side. However, the prime-to- p part of $\text{Gal}(k(\mu_{p^\infty})/k)$ has order $\ll p$. This contradiction proves the theorem in the present case.

Consider finally the case when there is a point x_n of order p^n with p fixed but n arbitrarily large. Without loss of generality, we may assume that μ_p is contained in k' . We shall prove again that $r = 1$. For some prime $\mathfrak{p} = \mathfrak{p}_1$ dividing p in K , the point x_n will have a \mathfrak{p} -component of large p -power order, and hence without loss of generality, we may assume that all the points x_n lie in $A[\mathfrak{p}^\infty]$ (the union of all the kernels of $\iota(\mathfrak{p}^v)$ for $v \rightarrow \infty$). In particular, the degrees $[k'(x_n):k']$ contain arbitrarily large powers of p , whence the fields $k'(x_n)$ contain arbitrarily large extensions $k'(\mu_{p^v})$. If $r \geq 2$ and \mathfrak{P}' is any prime ideal of k' dividing some prime $\mathfrak{p}_2, \dots, \mathfrak{p}_r$, then \mathfrak{P}' is unramified in $k'(x_n)$. But the ramification indices at all primes dividing p in k' tend to infinity as n tends to infinity. Hence again $r = 1$.

Now suppose that x_n has order \mathfrak{p}^n , meaning that \mathfrak{p}^n is the kernel of the map

$$\alpha \mapsto \iota(\alpha) x_n.$$

We shall prove that $k'(x_n) = k'(A[\mathfrak{p}^n])$. We have an isomorphism

$$\mathfrak{o}/\mathfrak{p}^n \approx \iota(\mathfrak{o}) x_n.$$

On the other hand, $A[\mathfrak{p}^n]$ is cyclic module over $\mathfrak{o}/\mathfrak{p}^n$, generated by an element z , so that $x_n = \iota(\alpha) z$ for some α . Then α must be a unit in the local ring of \mathfrak{o} at \mathfrak{p} , whence in fact

$$\iota(\mathfrak{o}) x = A[\mathfrak{p}^n].$$

This proves that $k'(x_n) = k'(A[\mathfrak{p}^n])$.

Using arbitrarily large n , we conclude that $k'(A[\mathfrak{p}^\infty])$ is contained in $k'(\mu_{p^\infty})$. But according to Kubota [Ku], the Galois group $\text{Gal}(k'(A[\mathfrak{p}^\infty])/k')$ is a Lie group of dimension ≥ 2 . Since the Galois group of the p -primary roots of unity is a Lie group of dimension 1, we have a contradiction, which concludes the proof.