Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FINITENESS THEOREMS IN GEOMETRIC CLASSFIELD THEORY
Autor: Katz, Nicholas M. / Lang, Serge
Kapitel: 0. Introduction
DOI: https://doi.org/10.5169/seals-51753

FINITENESS THEOREMS
IN GEOMETRIC CLASSFIELD THEORY

by Nicholas M. KATZ [1]) and Serge LANG [1])

(with an appendix by Kenneth A. RIBET)

0. INTRODUCTION

The geometric classfield theory of the 1950's was the principal precursor of the Grothendieck theory of the fundamental group developed in the early 1960's (cf. SGA I, Exp. X, 1.10). The problem was to understand the abelian unramified coverings of a variety X, or, as we would say today, to understand $\pi_1(X)^{ab}$. When X is "over" another variety S, the functoriality of π_1^{ab} gives a natural homomorphism.

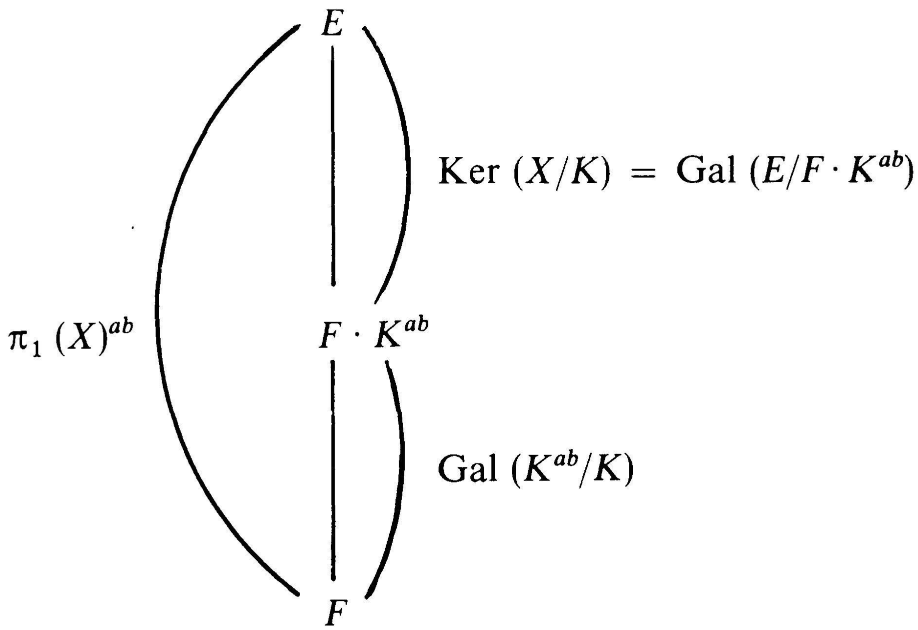
$$\pi_1(X)^{ab} \to \pi_1(S)^{ab}$$

whose kernel Ker (X/S) measures the extent to which the abelian coverings of X fail to "come from" abelian coverings of S.

In the language of the 1950's, we can make the problem "explicit" in terms of galois theory. Thus we consider the case when $S = \operatorname{Spec}(K)$, with K a field, and X a smooth and geometrically connected variety over K. Let F denote the function field of X, and denote by E/F the compositum, inside some fixed algebraic closure of F, of all finite abelian extensions E_i/F which are unramified over X in the sense that the normalization of X in E_i is finite etale over X. Then $\pi_1(X)^{ab}$ is "just" the galois group Gal (E/F).

Each finite extension L_i/K of K gives rise to a constant-field extension $F \cdot L_i$ over F which is abelian and unramified over X, so that if we denote by K^{ab} the maximal abelian extension of K, we have a diagram of fields and galois groups

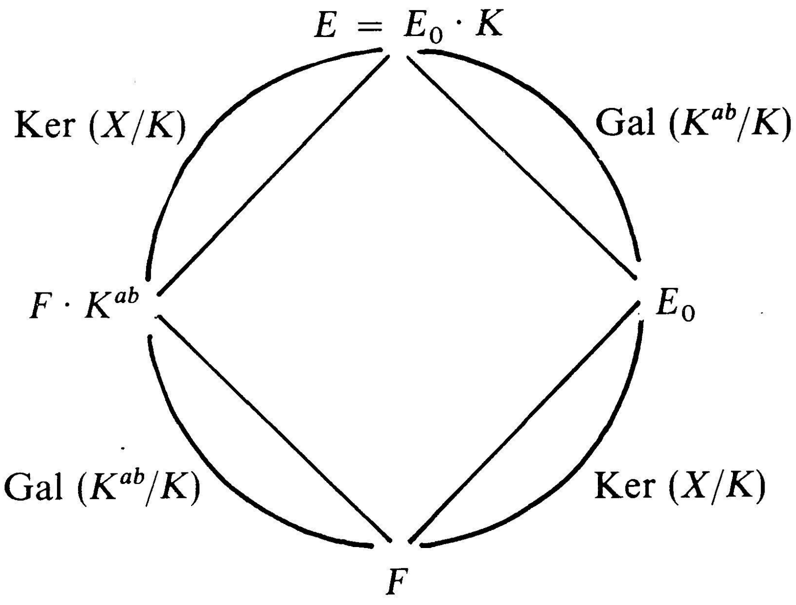
[1]) Supported by NSF grants.



and a corresponding exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}\,(X/K) & \to & \pi_1\,(X)^{ab} & \to & \mathrm{Gal}\,(K^{ab}/K) & \to & 0 \\ & & \| & & \| & & \| & & \\ & & \mathrm{Gal}\,(E/F\cdot K^{ab}) & & \mathrm{Gal}\,(E/F) & & \mathrm{Gal}\,(F\cdot K^{ab}/F) & & \end{array}$$

If we suppose further that X admits a K-rational point x_0, then we can "descend" the extension $E/(F\cdot K^{ab})$ to an extension E_0/F by the following device: we define E_0 to be the union of those finite abelian extensions E_i/F which are unramified over X, and such that the fibre over x_0 of the normalization X_i of X in E_i consists of deg (E_i/F) distinct K-rational points of X_i. Then E_0/F is a geometric extension, i.e. K is algebraically closed in E_0, and E is the compositum $E_0 \cdot K^{ab}$. Thus we have a diagram of fields and galois groups



and a corresponding splitting of the above exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}\,(X/K) & \to & \pi_1\,(X)^{ab} & \to & \mathrm{Gal}\,(K^{ab}/K) & \to & 0 \\ & & \| & & \| & & \| & & \\ & & \mathrm{Gal}\,(E_0/F) & & \mathrm{Gal}\,(E/F) & & \mathrm{Gal}\,(E/F_0) & & \end{array}$$

We will see that when K is a number field, i.e. a finite extension of **Q**, then the group Ker (X/K) is finite, or in other words the extension $E/(F \cdot K^{ab})$ (as well as the extension E_0/F when X has a K-rational point x_0) is *finite.*

Our main result is a finiteness theorem for the kernel group Ker (X/S) for a reasonably wide class of situations X/S which are sufficiently "of absolutely finite type" (cf. Theorems 1 & 2 for precise statements). When in addition we have *a priori* control of $\pi_1\,(S)^{ab}$, [as provided by global classfield theory when S is the (spectrum of) the ring of integers in a number field], or a systematic way of ignoring $\pi_1\,(S)^{ab}$ [e.g. if X/S has a section] we get "absolute" finiteness theorems (cf. Theorems 3, 4, 5 for precise statements). Following Deligne ([2], 1.3) and Grothendieck, we also give an application of our result to the theory of l-adic representations of fundamental groups of varieties over absolutely finitely generated fields (cf. Theorem 6 for a precise statement). In fact, it was this application, already exploited so spectacularly by Deligne in the case of varieties over finite fields, which aroused [resp. rearoused] our interest in the questions discussed here. For an application of these theorems to K-theory, we refer to recent work of Spencer Bloch [1] and A. H. Parshin [12].

The idea behind our proof is to reduce first to the case of an open curve over a field, by using Mike Artin's "good neighborhoods" and an elementary but useful exact homotopy sequence (cf. Preliminaries, Lemma 2). We then reduce to the case of an abelian variety over a field by using the theory of the generalized Jacobian. A standard specialization argument then reduces us to the case of an abelian variety over a finite field. In this case, we use Weil's form ([12], thm. 36) of the Lefschetz trace formula for abelian varieties to reduce our finiteness theorem to the fact that the number of rational points on an abelian variety over a finite field is finite and non-zero!

In explicating our results in the case of an abelian variety over a number field (cf. Section II, Remark 2), we were led to the conjecture that if A is an abelian variety over a number field k, and if $k\,(\mu)$ is the extension of k obtained by adjoining to k all roots of unity, then the torsion subgroup of A in $k\,(\mu)$ is finite. We shall prove this conjecture when A has complex multiplication. In an appendix, Ribet extends this result to a proof of the conjecture in general.