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APPLICATION OF TOPOLOGY TO PROBLEMS ON SUMS OF SQUARES

by Z. D. DAI, T. Y. LAM ¹⁾ and R. J. MILGRAM ¹⁾

If $n = 1, 2, 4$ or 8 , the classical n -square identities imply that the product of two sums of n squares in any commutative ring A is also a sum of n squares in A . On the other hand, by a classical theorem of Hurwitz [L, p. 137], one knows that the same statement cannot hold for other natural numbers n .

One can study the same problem over *fields* instead of over commutative rings. Here, the solution of the problem is also known, albeit somewhat different. According to a remarkable theorem of Pfister [P], if $n = 2^m$ is any 2-power, and if u, v are sums of n squares in a field F , then their product uv is also a sum of n squares in F . (This implies that the set of nonzero elements in F which are a sum of $n = 2^m$ squares in F is a group under multiplication.) On the other hand, Pfister has also shown that the above statement cannot hold for all fields if n is not of the form 2^m .

Back to sums of squares in commutative rings again, the above two paragraphs suggest that, in considering the multiplication problem, it is perhaps more reasonable to confine one's attention to *units* of a ring A which are sums of 2^m squares in A . Writing $n = 2^m$ and $U(A)$ for the group of units in A , one can ask:

- (*) If $u, v \in U(A)$ are sums of n squares in A ,
is $uv \in U(A)$ also a sum of n squares in A ?

This is equivalent to asking if the set of units in A which are a sum of $n = 2^m$ squares in A is a *group* under multiplication. This problem, first raised by R. Baeza, appeared as "Question 12" in Knebusch's collection [K₂] of open problems in the Proceedings of the Quadratic Form Conference in Kingston, Ontario in 1976. Generalizing the work of Pfister, Knebusch [K₁] has shown that (*) has an affirmative answer in case A is a (commutative) *semilocal* ring.

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In this note, we shall furnish the following solution to Baeza's problem:

THEOREM 1. *The answer to (*) is affirmative (for all commutative rings A) iff $n = 1, 2, 4$ or 8 .*

In view of the classical square identities mentioned before, we need only show the "only if" part of the theorem. The idea of the proof is to apply (*) in a "generic" setting, and then use suitable topological machinery to derive the conclusion $n = 1, 2, 4$ or 8 . The topological result needed here is Adams' famous theorem $[A_1]$ on the nonexistence of Hopf invariant one. Surprisingly, this algebraic application of Adams' Theorem, though reasonably straightforward, seems to have escaped the notice of both algebraists and topologists.

Let n be a natural number for which (*) holds for any commutative ring A . We shall prove that $n = 1, 2, 4$ or 8 . (In the following, we do not need to assume n to be a power of 2 to begin with, though this would follow from Pfister's theorem.)

Let A be the ring obtained by localizing the polynomial ring

$$\mathbf{R} [x_1, \dots, x_n, y_1, \dots, y_n]$$

at the multiplicative set generated by

$$u = x_1^2 + \dots + x_n^2 \quad \text{and} \quad v = y_1^2 + \dots + y_n^2.$$

Then, by (*), the unit $uv \in U(A)$ is a sum of n squares in A , say

$$uv = \left(\frac{f_1}{u^r v^s} \right)^2 + \dots + \left(\frac{f_n}{u^r v^s} \right)^2, \quad f_i \in \mathbf{R} [x, y].$$

Clearing denominators, we get a polynomial equation:

$$(1) \quad \begin{aligned} & x_1^2 + \dots + x_n^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1} \\ & = f_1(x, y)^2 + \dots + f_n(x, y)^2. \end{aligned}$$

Now we make the following key observation:

LEMMA 1. *Each $f_i(x, y)$ above is a "biform" in (x, y) , of bidegree $(2r+1, 2s+1)$ (i.e. viewing the y 's as constants, f_i is a form of degree $2r+1$ in x , and, viewing the x 's as constants, f_i is a form of degree $2s+1$ in y).*

Proof. View each f_i as a polynomial in x , and let f_{ij} denote its homogeneous component of degree j in x . We may write

$$f_i = f_{i,p} + f_{i,p+1} + \dots + f_{i,q} \quad (1 \leq i \leq n)$$

where p, q are independent of i . If $q > 2r + 1$, a comparison of terms of x -degree $2q$ on the two sides of (1) shows that

$$\sum_{i=1}^n f_{i,q}^2 = 0,$$

and hence $f_{i,q} = 0$ for all i . Similarly, if $p < 2r + 1$, we must have $f_{i,p} = 0$ for all i . Hence, f_i is a form in x of degree $2r + 1$. By symmetry, we infer that f_i is also a form in y of degree $2s + 1$. Q.E.D.

Now let x, y be points on the unit sphere S^{n-1} . The equation (1) above implies that the n -tuple

$$(f_1(x, y), \dots, f_n(x, y))$$

is also a point on S^{n-1} . Thus,

$$(x, y) \mapsto (f_1(x, y), \dots, f_n(x, y))$$

induces a polynomial (and hence continuous) mapping:

$$\mu: S^{n-1} \times S^{n-1} \rightarrow S^{n-1}.$$

Fix a base point $b \in S^{n-1}$. Then the compositions

$$S^{n-1} \rightarrow S^{n-1} \times \{b\} \xrightarrow{\mu} S^{n-1}$$

$$S^{n-1} \rightarrow \{b\} \times S^{n-1} \xrightarrow{\mu} S^{n-1}$$

are *odd* mappings, since each f_i has bidegree $(2r + 1, 2s + 1)$. By the theorem of Borsuk [B], these odd mappings from S^{n-1} to itself must have odd (topological) degrees, say, $2r' + 1$ and $2s' + 1$. Thus, the mapping μ has "type" $(2r' + 1, 2s' + 1)$ in the sense of Hopf [H₁].

Now by the Hopf Construction, the map μ induces a continuous map $\sigma: S^{2n-1} \rightarrow S^n$. Let

$$H: \pi_{2n-1}(S^n) \rightarrow \mathbf{Z}$$

be the Hopf invariant on the homotopy group $\pi_{2n-1}(S^n)$. According to Hopf [H₁, § 6], the homotopy class $[\sigma] \in \pi_{2n-1}(S^n)$ has Hopf invariant

$$H[\sigma] = \pm (2r' + 1)(2s' + 1),$$

which is an odd number. By Adams' theorem [A₁] on the nonexistence of Hopf invariant one (or odd Hopf invariant), one knows that this is possible only if $n = 1, 2, 4$ or 8 . This completes the proof of Theorem 1.

Adams' original solution of the Hopf invariant one problem took 85 pages, but there exists a proof using the powerful machinery of topological K -theory (cf. [A₆], [A₇, p. 137]) which, according to M. Atiyah, "can be written on a postcard". Thus, our Theorem 1 does admit a "short" proof. In fact, using K -theory, it is possible to obtain a more general version of Theorem 1. This will be deduced from the following topological statement:

THEOREM 2. *Let $\mu: S^{k-1} \times S^{n-1} \rightarrow S^{n-1}$ be a continuous mapping such that*

$$\mu(-x, y) = \mu(x, -y) = -\mu(x, y)$$

for all $x \in S^{k-1}$ and $y \in S^{n-1}$ (cf. [H₂]). Then $k \leq \rho(n)$, where ρ is the Hurwitz-Radon function.

(Recall that, if $n = 2^{4a+b} n_0$ where n_0 is odd and $b = 0, 1, 2$ or 3 , then, by definition, $\rho(n) = 8a + 2^b$.)

Before proving this theorem, let us first record several of its remarkable consequences in algebra. The first one is a result on real common zeros of biforms.

COROLLARY 1. *Let*

$$x = (x_1, \dots, x_k), y = (y_1, \dots, y_n).$$

Let $f_i(x, y)$ ($1 \leq i \leq n$) be biforms in (x, y) of odd bidegrees $(2r_i + 1, 2s_i + 1)$. If $k > \rho(n)$, then the real loci of $f_i = 0$ in the multiprojective space $\mathbf{RP}^{k-1} \times \mathbf{RP}^{n-1}$ have a common point.

Proof. If otherwise, we would have a mapping μ as in Theorem 2 defined by

$$\mu(x, y) = (f_1(x, y)/g(x, y), \dots, f_n(x, y)/g(x, y))$$

where $g(x, y) = (\sum f_i(x, y)^2)^{1/2}$.

COROLLARY 2. *Let*

$$F(x, y) = F(x_1, \dots, x_k; y_1, \dots, y_n)$$

be a biform of bidegree (d, e) where d, e are not multiples of 4, and $k > \rho(n)$. Suppose that

$$F(x_o, y_o) = 0 \ (x_o \in \mathbf{R}^k, y_o \in \mathbf{R}^n) \Rightarrow x_o = 0 \text{ or } y_o = 0.$$

Then F cannot be a sum of n squares in $\mathbf{R}[x, y]$.

Proof. This is clear from the above Corollary and the argument given in Lemma 1.

The next corollary may be viewed as a nonlinear generalization of the classical Hurwitz-Radon Theorem [L, p. 137]:

COROLLARY 3. For

$$x = (x_1, \dots, x_k), y = (y_1, \dots, y_n)$$

and fixed integers $r, s \geq 0$, the following statements are equivalent:

- (1) $(x_1^2 + \dots + x_k^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1}$ is a sum of n squares in $\mathbf{R}[x, y]$;
- (2) $(x_1^2 + \dots + x_k^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1}$ is a sum of n squares in $\mathbf{Z}[x, y]$;
- (3) $k \leq \rho(n)$.

Proof. (2) \Rightarrow (1) is obvious.

(1) \Rightarrow (3) follows from Corollary 2.

(3) \Rightarrow (2): It is enough to prove (2) for $r = s = 0$. This follows from $[G_1]$ or $[G_2]$.

For a commutative ring A , let $S_m(A)$ denote the set of sums of m squares in A , and let $US_m(A) = U(A) \cap S_m(A)$.

COROLLARY 4. For fixed integers k and n , the following statements are equivalent:

- (1) For any commutative \mathbf{R} -algebra A , $US_k(A) \cdot US_n(A) \subseteq US_n(A)$;
- (2) For any commutative ring A , $S_k(A) \cdot S_n(A) \subseteq S_n(A)$;
- (3) $k \leq \rho(n)$.

Proof. This is clear from Corollary 3 and the localization argument we have given before. (Note that Theorem 1 is a special case of this Corollary since it is well-known that $n \leq \rho(n)$ iff $n = 1, 2, 4$ or 8 .)

We shall now begin the proof of Theorem 2, using tools from K -theory, especially Adams' work on the J -homomorphism. For any finite CW -complex X , let $KO(X)$ denote the K -group of virtual real vector bundles over X , and

$\widetilde{KO}(X)$ the reduced K -group (modulo trivial bundles). Let $J(X)$ denote the group of stable fiber homotopy equivalence classes of virtual sphere bundles over X , and $\tilde{J}(X)$ the reduced J -group. The canonical J -homomorphism $J: KO(X) \rightarrow J(X)$ induces a homomorphism $\tilde{J}: \widetilde{KO}(X) \rightarrow \tilde{J}(X)$. We shall use Adams' results in the following form (see [A₂, (7.4)], [A₄, (3.5)] and [A₅]):

ADAMS' THEOREM. For $X = \mathbf{RP}^m$, \tilde{J} is an isomorphism $\widetilde{KO}(X) \cong \tilde{J}(X)$. The group $\widetilde{KO}(X)$ is cyclic of order $2^{\phi(m)}$ where $\phi(m)$ is the number of positive integers $\leq m$ which are congruent to 0, 1, 2 or 4 (mod 8). A generator for $\widetilde{KO}(X)$ is given by the canonical line bundle ξ_m over \mathbf{RP}^m .

On the product $S^{k-1} \times S^{n-1}$, we have an involution defined by

$$T(x, y) = (-x, -y);$$

let E be the quotient space $S^{k-1} \times S^{n-1}/T$. We have an $(n-1)$ -sphere bundle $\eta: E \rightarrow \mathbf{RP}^{k-1}$: this is the associated sphere bundle of the Whitney sum $n \cdot \xi_{k-1}$. Note that E has an involution $\tau(x, y) = (x, -y)$ which on each fiber is the antipodal map.

Assume that we have a continuous map

$$\mu: S^{k-1} \times S^{n-1} \rightarrow S^{n-1},$$

as in Theorem 2. Then μ induces a map $\bar{\mu}: E \rightarrow S^{n-1}$ which is equivariant with respect to the involution τ on E and the antipodal map on S^{n-1} . We have a fiber map

$$(\eta, \bar{\mu}): E \rightarrow \mathbf{RP}^{k-1} \times S^{n-1} \quad (\text{trivial bundle over } \mathbf{RP}^{k-1})$$

which (by the theorem of Borsuk again) has odd degree d on each fiber. By the "mod d -Dold Theorem" [A₃, (1.1)], there exists an integer $e \geq 0$ such that $d^e \cdot \eta$ is fiber homotopy equivalent to a trivial bundle. Since $\tilde{J}(\mathbf{RP}^{k-1})$ is 2-primary, this implies that $\eta = 0$ in $\tilde{J}(\mathbf{RP}^{k-1})$. Pulling back to $\widetilde{KO}(\mathbf{RP}^{k-1})$, we have $n \cdot \xi_{k-1} = 0$ in $\widetilde{KO}(\mathbf{RP}^{k-1})$, so by Adams' Theorem, n is divisible by $2^{\phi(k-1)}$. Let $n = 2^{4a+b} n_0$ where n_0 is odd and $b = 0, 1, 2$ or 3 . If $k > \rho(n) = 8a + 2^b$, then

$$\phi(k-1) \geq \phi(8a+2^b) = 4a + b + 1,$$

contradicting $2^{\phi(k-1)} \mid n$. Therefore, we have $k \leq \rho(n)$ as desired.

In a recent communication to us, I. M. James has suggested a similar proof of Theorem 2. He points out that a more general discussion of similar structures from which Theorem 2 follows may be found in [W] and in [J, Sec. 7].

REFERENCES

- [A₁] ADAMS, J. On the nonexistence of elements of Hopf invariant one. *Annals of Math.* 72 (1960), 20-104.
- [A₂] ——— Vector fields on spheres. *Annals of Math.* 75 (1962), 603-632.
- [A₃] ——— On the groups $J(X)$, I. *Topology* 2 (1963), 181-195.
- [A₄] ——— On the groups $J(X)$, II. *Topology* 3 (1965), 137-171.
- [A₅] ——— On the groups $J(X)$, III. *Topology* 3 (1965), 193-222.
- [A₆] ADAMS, J. and M. ATIYAH. K -theory and the Hopf invariant. *Quart. J. Math.* 17 (1966), 31-38.
- [A₇] ATIYAH, M. F. *K-Theory*. W. A. Benjamin, Inc., Reading, Massachusetts, 1967.
- [B] BORSUK, K. Drei Sätze über die n -dimensionale Euklidische Sphäre. *Fund. Math.* 20 (1933), 177-190.
- [G₁] GABEL, M. R. Generic orthogonal stably free projectives. *J. Algebra* 29 (1974), 477-488.
- [G₂] GERAMITA, A. and N. PULLMAN. A theorem of Hurwitz and Radon and orthogonal projective modules. *Proc. Amer. Math. Soc.* 42 (1974), 51-56.
- [H₁] HOPF, H. Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension. *Fund. Math.* 25 (1935), 427-440.
- [H₂] ——— Ein topologischer Beitrag zur reellen Algebra. *Comment. Math. Helv.* 13 (1940), 219-239.
- [J] JAMES, I. M. *The Topology of Stiefel Manifolds*. London Math. Soc. Lecture Notes Series, Vol. 24, Cambridge University Press, 1976.
- [K₁] KNEBUSCH, M. Runde Formen über semilokalen Ringen. *Math. Ann.* 193 (1971), 21-34.
- [K₂] ——— Some open problems. In: *Proc. Quadratic Form Conference* (ed. G. Orzech), Queen's Papers in Pure and Applied Math., Vol. 55, pp. 361-370, Queen's University, Kingston, Ontario, 1977.
- [L] LAM, T. Y. *The Algebraic Theory of Quadratic Forms*. W. A. Benjamin, Inc., 1973.
- [P] PFISTER, A. Multiplikative quadratische Formen. *Arch. Math.* 16 (1965), 363-370.
- [W] WOODWARD, L. M. Vector fields on spheres and a generalization. *Quart. J. Math.* 24 (1973), 357-366.

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