

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 27 (1981)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** APPLICATION OF TOPOLOGY TO PROBLEMS ON SUMS OF SQUARES  
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**DOI:** <https://doi.org/10.5169/seals-51752>

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# APPLICATION OF TOPOLOGY TO PROBLEMS ON SUMS OF SQUARES

by Z. D. DAI, T. Y. LAM <sup>1)</sup> and R. J. MILGRAM <sup>1)</sup>

If  $n = 1, 2, 4$  or  $8$ , the classical  $n$ -square identities imply that the product of two sums of  $n$  squares in any commutative ring  $A$  is also a sum of  $n$  squares in  $A$ . On the other hand, by a classical theorem of Hurwitz [L, p. 137], one knows that the same statement cannot hold for other natural numbers  $n$ .

One can study the same problem over *fields* instead of over commutative rings. Here, the solution of the problem is also known, albeit somewhat different. According to a remarkable theorem of Pfister [P], if  $n = 2^m$  is *any* 2-power, and if  $u, v$  are sums of  $n$  squares in a field  $F$ , then their product  $uv$  is also a sum of  $n$  squares in  $F$ . (This implies that the set of nonzero elements in  $F$  which are a sum of  $n = 2^m$  squares in  $F$  is a group under multiplication.) On the other hand, Pfister has also shown that the above statement cannot hold for all fields if  $n$  is not of the form  $2^m$ .

Back to sums of squares in commutative rings again, the above two paragraphs suggest that, in considering the multiplication problem, it is perhaps more reasonable to confine one's attention to *units* of a ring  $A$  which are sums of  $2^m$  squares in  $A$ . Writing  $n = 2^m$  and  $U(A)$  for the group of units in  $A$ , one can ask:

(\*) If  $u, v \in U(A)$  are sums of  $n$  squares in  $A$ ,  
is  $uv \in U(A)$  also a sum of  $n$  squares in  $A$ ?

This is equivalent to asking if the set of units in  $A$  which are a sum of  $n = 2^m$  squares in  $A$  is a *group* under multiplication. This problem, first raised by R. Baeza, appeared as "Question 12" in Knebusch's collection [K<sub>2</sub>] of open problems in the Proceedings of the Quadratic Form Conference in Kingston, Ontario in 1976. Generalizing the work of Pfister, Knebusch [K<sub>1</sub>] has shown that (\*) has an affirmative answer in case  $A$  is a (commutative) *semilocal* ring.

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<sup>1)</sup> Supported in part by NSF.

In this note, we shall furnish the following solution to Baeza's problem:

**THEOREM 1.** *The answer to (\*) is affirmative (for all commutative rings  $A$ ) iff  $n = 1, 2, 4$  or  $8$ .*

In view of the classical square identities mentioned before, we need only show the "only if" part of the theorem. The idea of the proof is to apply (\*) in a "generic" setting, and then use suitable topological machinery to derive the conclusion  $n = 1, 2, 4$  or  $8$ . The topological result needed here is Adams' famous theorem  $[A_1]$  on the nonexistence of Hopf invariant one. Surprisingly, this algebraic application of Adams' Theorem, though reasonably straightforward, seems to have escaped the notice of both algebraists and topologists.

Let  $n$  be a natural number for which (\*) holds for any commutative ring  $A$ . We shall prove that  $n = 1, 2, 4$  or  $8$ . (In the following, we do not need to assume  $n$  to be a power of 2 to begin with, though this would follow from Pfister's theorem.)

Let  $A$  be the ring obtained by localizing the polynomial ring

$$\mathbf{R} [x_1, \dots, x_n, y_1, \dots, y_n]$$

at the multiplicative set generated by

$$u = x_1^2 + \dots + x_n^2 \quad \text{and} \quad v = y_1^2 + \dots + y_n^2.$$

Then, by (\*), the unit  $uv \in U(A)$  is a sum of  $n$  squares in  $A$ , say

$$uv = \left( \frac{f_1}{u^r v^s} \right)^2 + \dots + \left( \frac{f_n}{u^r v^s} \right)^2, \quad f_i \in \mathbf{R} [x, y].$$

Clearing denominators, we get a polynomial equation:

$$(1) \quad \begin{aligned} & x_1^2 + \dots + x_n^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1} \\ &= f_1(x, y)^2 + \dots + f_n(x, y)^2. \end{aligned}$$

Now we make the following key observation:

**LEMMA 1.** *Each  $f_i(x, y)$  above is a "biform" in  $(x, y)$ , of bidegree  $(2r+1, 2s+1)$  (i.e. viewing the  $y$ 's as constants,  $f_i$  is a form of degree  $2r+1$  in  $x$ , and, viewing the  $x$ 's as constants,  $f_i$  is a form of degree  $2s+1$  in  $y$ ).*

*Proof.* View each  $f_i$  as a polynomial in  $x$ , and let  $f_{ij}$  denote its homogeneous component of degree  $j$  in  $x$ . We may write

$$f_i = f_{i,p} + f_{i,p+1} + \dots + f_{i,q} \quad (1 \leq i \leq n)$$

where  $p, q$  are independent of  $i$ . If  $q > 2r+1$ , a comparison of terms of  $x$ -degree  $2q$  on the two sides of (1) shows that

$$\sum_{i=1}^n f_{i,q}^2 = 0,$$

and hence  $f_{i,q} = 0$  for all  $i$ . Similarly, if  $p < 2r+1$ , we must have  $f_{i,p} = 0$  for all  $i$ . Hence,  $f_i$  is a form in  $x$  of degree  $2r+1$ . By symmetry, we infer that  $f_i$  is also a form in  $y$  of degree  $2s+1$ . Q.E.D.

Now let  $x, y$  be points on the unit sphere  $S^{n-1}$ . The equation (1) above implies that the  $n$ -tuple

$$(f_1(x, y), \dots, f_n(x, y))$$

is also a point on  $S^{n-1}$ . Thus,

$$(x, y) \mapsto (f_1(x, y), \dots, f_n(x, y))$$

induces a polynomial (and hence continuous) mapping:

$$\mu: S^{n-1} \times S^{n-1} \rightarrow S^{n-1}.$$

Fix a base point  $b \in S^{n-1}$ . Then the compositions

$$S^{n-1} \rightarrow S^{n-1} \times \{b\} \xrightarrow{\mu} S^{n-1}$$

$$S^{n-1} \rightarrow \{b\} \times S^{n-1} \xrightarrow{\mu} S^{n-1}$$

are *odd* mappings, since each  $f_i$  has bidegree  $(2r+1, 2s+1)$ . By the theorem of Borsuk [B], these odd mappings from  $S^{n-1}$  to itself must have odd (topological) degrees, say,  $2r'+1$  and  $2s'+1$ . Thus, the mapping  $\mu$  has "type"  $(2r'+1, 2s'+1)$  in the sense of Hopf [H<sub>1</sub>].

Now by the Hopf Construction, the map  $\mu$  induces a continuous map  $\sigma: S^{2n-1} \rightarrow S^n$ . Let

$$H: \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$$

be the Hopf invariant on the homotopy group  $\pi_{2n-1}(S^n)$ . According to Hopf [H<sub>1</sub>, § 6], the homotopy class  $[\sigma] \in \pi_{2n-1}(S^n)$  has Hopf invariant

$$H[\sigma] = \pm (2r'+1)(2s'+1),$$

which is an odd number. By Adams' theorem  $[A_1]$  on the nonexistence of Hopf invariant one (or odd Hopf invariant), one knows that this is possible only if  $n = 1, 2, 4$  or  $8$ . This completes the proof of Theorem 1.

Adams' original solution of the Hopf invariant one problem took 85 pages, but there exists a proof using the powerful machinery of topological  $K$ -theory (cf.  $[A_6]$ ,  $[A_7]$ , p. 137) which, according to M. Atiyah, "can be written on a postcard". Thus, our Theorem 1 does admit a "short" proof. In fact, using  $K$ -theory, it is possible to obtain a more general version of Theorem 1. This will be deduced from the following topological statement:

**THEOREM 2.** *Let  $\mu: S^{k-1} \times S^{n-1} \rightarrow S^{n-1}$  be a continuous mapping such that*

$$\mu(-x, y) = \mu(x, -y) = -\mu(x, y)$$

*for all  $x \in S^{k-1}$  and  $y \in S^{n-1}$  (cf.  $[H_2]$ ). Then  $k \leq \rho(n)$ , where  $\rho$  is the Hurwitz-Radon function.*

(Recall that, if  $n = 2^{4a+b} n_0$  where  $n_0$  is odd and  $b = 0, 1, 2$  or  $3$ , then, by definition,  $\rho(n) = 8a + 2^b$ .)

Before proving this theorem, let us first record several of its remarkable consequences in algebra. The first one is a result on real common zeros of biforms.

**COROLLARY 1.** *Let*

$$x = (x_1, \dots, x_k), y = (y_1, \dots, y_n).$$

*Let  $f_i(x, y)$  ( $1 \leq i \leq n$ ) be biforms in  $(x, y)$  of odd bidegrees  $(2r_i + 1, 2s_i + 1)$ . If  $k > \rho(n)$ , then the real loci of  $f_i = 0$  in the multiprojective space  $\mathbf{RP}^{k-1} \times \mathbf{RP}^{n-1}$  have a common point.*

*Proof.* If otherwise, we would have a mapping  $\mu$  as in Theorem 2 defined by

$$\mu(x, y) = (f_1(x, y)/g(x, y), \dots, f_n(x, y)/g(x, y))$$

where  $g(x, y) = (\sum f_i(x, y)^2)^{1/2}$ .

**COROLLARY 2.** *Let*

$$F(x, y) = F(x_1, \dots, x_k; y_1, \dots, y_n)$$

*be a biform of bidegree  $(d, e)$  where  $d, e$  are not multiples of 4, and  $k > \rho(n)$ . Suppose that*

$$F(x_o, y_o) = 0 \ (x_o \in \mathbf{R}^k, y_o \in \mathbf{R}^n) \Rightarrow x_o = 0 \text{ or } y_o = 0.$$

Then  $F$  cannot be a sum of  $n$  squares in  $\mathbf{R}[x, y]$ .

*Proof.* This is clear from the above Corollary and the argument given in Lemma 1.

The next corollary may be viewed as a nonlinear generalization of the classical Hurwitz-Radon Theorem [L, p. 137]:

COROLLARY 3. For

$$x = (x_1, \dots, x_k), y = (y_1, \dots, y_n)$$

and fixed integers  $r, s \geq 0$ , the following statements are equivalent:

- (1)  $(x_1^2 + \dots + x_k^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1}$  is a sum of  $n$  squares in  $\mathbf{R}[x, y]$ ;
- (2)  $(x_1^2 + \dots + x_k^2)^{2r+1} (y_1^2 + \dots + y_n^2)^{2s+1}$  is a sum of  $n$  squares in  $\mathbf{Z}[x, y]$ ;
- (3)  $k \leq \rho(n)$ .

*Proof.* (2)  $\Rightarrow$  (1) is obvious.

(1)  $\Rightarrow$  (3) follows from Corollary 2.

(3)  $\Rightarrow$  (2): It is enough to prove (2) for  $r = s = 0$ . This follows from  $[G_1]$  or  $[G_2]$ .

For a commutative ring  $A$ , let  $S_m(A)$  denote the set of sums of  $m$  squares in  $A$ , and let  $US_m(A) = U(A) \cap S_m(A)$ .

COROLLARY 4. For fixed integers  $k$  and  $n$ , the following statements are equivalent:

- (1) For any commutative  $\mathbf{R}$ -algebra  $A$ ,  $US_k(A) \cdot US_n(A) \subseteq US_n(A)$ ;
- (2) For any commutative ring  $A$ ,  $S_k(A) \cdot S_n(A) \subseteq S_n(A)$ ;
- (3)  $k \leq \rho(n)$ .

*Proof.* This is clear from Corollary 3 and the localization argument we have given before. (Note that Theorem 1 is a special case of this Corollary since it is well-known that  $n \leq \rho(n)$  iff  $n = 1, 2, 4$  or  $8$ .)

We shall now begin the proof of Theorem 2, using tools from  $K$ -theory, especially Adams' work on the  $J$ -homomorphism. For any finite  $CW$ -complex  $X$ , let  $KO(X)$  denote the  $K$ -group of virtual real vector bundles over  $X$ , and

$\widetilde{KO}(X)$  the reduced  $K$ -group (modulo trivial bundles). Let  $J(X)$  denote the group of stable fiber homotopy equivalence classes of virtual sphere bundles over  $X$ , and  $\tilde{J}(X)$  the reduced  $J$ -group. The canonical  $J$ -homomorphism  $J: KO(X) \rightarrow J(X)$  induces a homomorphism  $\tilde{J}: \widetilde{KO}(X) \rightarrow \tilde{J}(X)$ . We shall use Adams' results in the following form (see  $[A_2, (7.4)]$ ,  $[A_4, (3.5)]$  and  $[A_5]$ ):

**ADAMS' THEOREM.** For  $X = \mathbf{RP}^m$ ,  $\tilde{J}$  is an isomorphism  $\widetilde{KO}(X) \cong \tilde{J}(X)$ . The group  $\widetilde{KO}(X)$  is cyclic of order  $2^{\phi(m)}$  where  $\phi(m)$  is the number of positive integers  $\leq m$  which are congruent to 0, 1, 2 or 4 (mod 8). A generator for  $\widetilde{KO}(X)$  is given by the canonical line bundle  $\xi_m$  over  $\mathbf{RP}^m$ .

On the product  $S^{k-1} \times S^{n-1}$ , we have an involution defined by

$$T(x, y) = (-x, -y);$$

let  $E$  be the quotient space  $S^{k-1} \times S^{n-1}/T$ . We have an  $(n-1)$ -sphere bundle  $\eta: E \rightarrow \mathbf{RP}^{k-1}$ : this is the associated sphere bundle of the Whitney sum  $n \cdot \xi_{k-1}$ . Note that  $E$  has an involution  $\tau(x, y) = (x, -y)$  which on each fiber is the antipodal map.

Assume that we have a continuous map

$$\mu: S^{k-1} \times S^{n-1} \rightarrow S^{n-1},$$

as in Theorem 2. Then  $\mu$  induces a map  $\bar{\mu}: E \rightarrow S^{n-1}$  which is equivariant with respect to the involution  $\tau$  on  $E$  and the antipodal map on  $S^{n-1}$ . We have a fiber map

$$(\eta, \bar{\mu}): E \rightarrow \mathbf{RP}^{k-1} \times S^{n-1} \quad (\text{trivial bundle over } \mathbf{RP}^{k-1})$$

which (by the theorem of Borsuk again) has odd degree  $d$  on each fiber. By the "mod  $d$ -Dold Theorem"  $[A_3, (1.1)]$ , there exists an integer  $e \geq 0$  such that  $d^e \cdot \eta$  is fiber homotopy equivalent to a trivial bundle. Since  $\tilde{J}(\mathbf{RP}^{k-1})$  is 2-primary, this implies that  $\eta = 0$  in  $\tilde{J}(\mathbf{RP}^{k-1})$ . Pulling back to  $\widetilde{KO}(\mathbf{RP}^{k-1})$ , we have  $n \cdot \xi_{k-1} = 0$  in  $\widetilde{KO}(\mathbf{RP}^{k-1})$ , so by Adams' Theorem,  $n$  is divisible by  $2^{\phi(k-1)}$ . Let  $n = 2^{4a+b} n_0$  where  $n_0$  is odd and  $b = 0, 1, 2$  or  $3$ . If  $k > \rho(n) = 8a + 2^b$ , then

$$\phi(k-1) \geq \phi(8a+2^b) = 4a + b + 1,$$

contradicting  $2^{\phi(k-1)} \mid n$ . Therefore, we have  $k \leq \rho(n)$  as desired.

In a recent communication to us, I. M. James has suggested a similar proof of Theorem 2. He points out that a more general discussion of similar structures from which Theorem 2 follows may be found in  $[W]$  and in  $[J, \text{Sec. 7}]$ .

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(Reçu le 30 novembre 1980)

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