Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE

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DOI: https://doi.org/10.5169/seals-51751

§4. THE VARIATION OF HODGE STRUCTURE ASSOCIATED TO (ρ, V)

Under the assumption that V is a real representation of G, the Hodge theory for $H^n(S, \mathbf{V})$ fits, perhaps surprisingly, into the more general framework of [11]. Associated to the irreducible representation (ρ, V) of G, there is a homogeneous variation of Hodge structure on the Hermitian symmetric space M, which we shall now define and analyze. The notion turns out to be a variant of ideas in [16, §1].

We recall some basic definitions from Hodge theory. We will use some of the same symbols that were employed in the preceding sections, in a more general context, so that the passage from representation theory to Hodge theory will be clearly laid out. Thus, let $V_{\mathbf{R}}$ be a finite-dimensional real vector space, V its complexification.

(4.1) *Definition.* A *Hodge structure of weight* m on V is a direct sum decomposition

$$V = \bigoplus_{\substack{p,\, q \in \mathbf{Z} \\ p+q=m}} H^{p,q} .$$

such that $\overline{H^{p,q}} = H^{q,p}$.

Let $\mathfrak{C}$ be the *Weil operator* of the Hodge structure, defined as the direct sum of the scalar operators i^{p-q} on $H^{p,q}$.

(4.2) *Definition.* A *polarization* of a Hodge structure is a bilinear form $\beta(v, w)$ on $V_{\mathbf{R}}$ such that $\beta(\mathfrak{C}v, \bar{v}) > 0$ whenever $v \neq 0$, and the $H^{p,q}$ spaces are orthogonal with respect to the Hermitian extension of β to V.

(4.3) *Remark.* The definition (4.2) implies the condition usually imposed on β: that it be symmetric if m is even, skew if m is odd.

Ordinarily, the primary example of a polarized Hodge structure of weight m is the cohomology group $H^m(M, \mathbf{C})$ of a compact Kähler manifold M, with a polarization built from cup-product.

(4.4) *Definition.* The *Hodge filtration*

$$\ldots \supset F^r \supset F^{r+1} \supset \ldots$$

is defined by $F^r = \bigoplus_{p \geq r} H^{p,q}$.

One recovers $H^{p,q}$ as $F^p \cap \overline{F^q}$.

Let S be a complex manifold. We will give the definition of a *polarizable* (real) *variation of Hodge structure of weight* m in terms of the universal covering $\pi: M \to S$. Let $\Gamma = \pi_1(S)$, viewed as the group of deck transformations of M. Let $(\rho, V_{\mathbf{R}})$ be a finite dimensional representation of Γ (the *monodromy*). Take $M \times V$: if we place the usual topology on V, we have a vector bundle $\mathscr{V}$; placing the discrete topology on V, we get a constant sheaf $\mathbf{V}$. We require:

(4.5) i) For each $x \in M$, there is a Hodge structure of weight m:

$$V = \oplus H_x^{p,q}.$$

ii) For each (p, q),

$$\mathscr{H}^{p,q} = \coprod_{x \in M} (\{x\} \times H_x^{p,q})$$

forms a C^∞ sub-bundle of $\mathscr{V}$.

iii) For each r,

$$\mathscr{F}^r = \coprod_{x \in M} (\{x\} \times F_x^r) \qquad (F_x^r = \bigoplus_{p \geqslant r} H_x^{p,q})$$

forms a holomorphic sub-bundle of $\mathscr{V}$.

iv) If σ is a local holomorphic section of $\mathscr{F}^r$, and X is a local holomorphic vectorfield on M, then $X\sigma$ is a section of $\mathscr{F}^{r-1}$.

v) There exists a *flat* bilinear form which polarizes the Hodge structure for each x.

(In order to pass this data down to S, we add)

vi) If $\gamma \in \Gamma$, $\rho(\gamma) H_x^{p,q} = H_{\gamma x}^{p,q}$.

We can then take quotients of $\mathscr{V}$, $\mathscr{F}^r$ and $\mathbf{V}$ by Γ to obtain objects on S, which will also be denoted $\mathscr{V}$, $\mathscr{F}^r$ and $\mathbf{V}$, hopefully without confusion.

Ordinarily, the primary example of such $\mathbf{V}$ underlying polarized variations of Hodge structure are the systems $R^m f_* \mathbf{C}$ of m-dimensional cohomology along the fibers for families of Kähler manifolds $f: \mathfrak{M} \to S$.

It is useful to relax the conditions of (4.5). Following a suggestion of P. Deligne, we give:

(4.6) *Definition.* Let (ρ, V) be a finite dimensional representation of Γ (not necessarily real) and $\mathbf{V}$ the resulting locally constant sheaf. By a *complex variation of Hodge structure of weight* m, we mean the collection of data described in (4.5) with the following modifications:

a) For (i), we drop the requirement that $\overline{H_x^{p,q}} = H_x^{q,p}$, for it may be inconsistent with (vi). (One might call the resulting decomposition in (4.1) a *complex Hodge structure*.)

b) To (iii), we add for each s,

$$\bar{\mathscr{F}}^s = \coprod_{x \in M} (\{x\} \times \bar{F}_x^s) \qquad (\bar{F}_x^s = \bigoplus_{q \geqslant s} H_x^{p,q})$$

forms an anti-holomorphic sub-bundle of $\mathscr{V}$.

c) To (iv), add: if σ is a local anti-holomorphic section of $\bar{\mathscr{F}}^r$ and $\bar{X}$ is a local anti-holomorphic vector field on M, then $\bar{X}\sigma$ is a section of $\bar{\mathscr{F}}^{r-1}$.

d) Replace (v) by: there is a flat sequilinear pairing

$$\bar{\beta}: \mathbf{V} \times \mathbf{V} \to \mathbf{C},$$

such that $\bar{\beta}(\mathfrak{C}v, v) > 0$ whenever $v \neq 0$.

(4.7) *Remark.* When V is real, and one has a (real) variation of Hodge structure, then (b) and (c) of (4.6) are automatic, and $\bar{\beta}$ is given by

$$\bar{\beta}(v, w) = \beta(v, \bar{w}).$$

Let now (ρ, V) be an irreducible representation of the Lie group G. Then we may write, according to (1.5) and (1.7)

$$V = \bigoplus_{s=0}^{m} V\langle\lambda - s\mu\rangle, \tag{4.8}$$

where χ_λ is the highest character occurring in (ρ, V). We convert (4.8) into a complex variation of Hodge structure of weight m on $S = \Gamma\backslash M$ by first setting

$$H_0^{p,q} = V\langle\lambda - p\mu\rangle \tag{4.9}$$

if $p, q \geqslant 0$ and $p + q = m$; we then define

$$H_x^{p,q} = \rho(g) H_0^{p,q} \quad \text{if} \quad gx_0 \in M. \tag{4.10}$$

For obvious reasons, we will call this a *locally homogeneous* variation of Hodge structure. It is real whenever (ρ, V) is.

We must verify that the conditions of (4.6) hold; we follow the numbering in (4.5). Because of (1.7, i), the space $H_x^{p,q}$ is well-defined (in the real case, use also (1.7, iii)), so (i) is satisfied. Properties (ii) and (vi) pose no difficulty. We get (iii) from the fact that $F_0^r = \bigoplus_{p \geqslant r} H_0^{p,q}$ (resp. $\bar{F}_0^s = \bigoplus_{q \geqslant s} H_0^{p,q}$) is a $K_{\mathbf{C}}P^-$ (resp.

$P^+K_{\mathbf{C}}$)-invariant subspace of V, and (iv) from the fact that $\mathfrak{p}^+F_0^r \subset F_0^{r-1}$ (resp. $\mathfrak{p}^-\bar{F}_0^s \subset \bar{F}_0^{s-1}$); both assertions follow from (1.7, i) and (4.9).

The flat polarization (4.6, d) ((4.5, v) in the real case) is provided by the admissible inner product T (1.9). Let $\mathfrak{C}_0$ denote the Weil operator of (4.8). Then

$$T(v, w) = \bar{\beta}(\mathfrak{C}_0 v, w) \qquad \text{if we put} \qquad \bar{\beta}(v, w) = T(\mathfrak{C}_0^{-1} v, w)\,. \tag{4.11}$$

We assert that $\bar{\beta}$ is G-invariant (with G acting by $\bar{\rho}$ on the second entry). For this, we need only apply (1.9) to see that

$$\bar{\beta}(\rho(X)\, v, w) + \bar{\beta}(v, \overline{\rho(X)}\, w) = 0$$

for all $X \in \mathfrak{g}_{\mathbf{C}}$, $v, w \in V$. (In the real case, we are displaying the self-contragredience of ρ.) That $\bar{\beta}$ determines a polarization now follows by homogeneity. This completes our verification.

Note that at $gx_0 \in M$,

$$\begin{aligned} \bar{\beta}(\mathfrak{C}_{gx_0} v, w) &= \bar{\beta}(\rho(g)\, \mathfrak{C}_0\, \rho(g)^{-1}\, v, w) \\ &= \bar{\beta}(\mathfrak{C}_0\, \rho(g)^{-1}\, v, \overline{\rho(g)^{-1}}\, w) \\ &= T(\rho(g)^{-1}\, v, \overline{\rho(g)^{-1}}\, w)\,, \end{aligned} \tag{4.12}$$

so the "Hodge metric" coincides with the one given in (2.8). Also, $\eta \in \mathscr{A}^n(S, \mathbf{V})$ takes its values in $\mathscr{H}^{p,q}$ if and only if $\tilde{\eta} \in \mathscr{A}^n(\Gamma\backslash G) \otimes H_0^{p,q}$.

§5. Hodge theory for $H^n(\Gamma; \rho, V)$, from the variation of Hodge structure

In this Section, we will review the general Hodge theory for locally constant sheaves $\mathbf{V}$ underlying polarizable variations of Hodge structure. After that, we will insert the construction of (4.9) into the general framework and draw special conclusions about this case. There are both local considerations and global results. The latter follow "automatically" only when S is compact, in which case they are due to Deligne (see [11, §§1-2]). The global results generalize to non-compact quotients of finite volume for $G = SL(2, \mathbf{R})$ [11, §§7, 12], and hopefully we will soon be able to handle $G = SU(n, 1)$. We should view the compact case as providing formal guidelines for a general theory.