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### §3. THE COHOMOLOGY GROUPS $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups  $H^n(\Gamma; \rho, V)$  for a finite-dimensional representation  $(\rho, V)$  of  $G$ , which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since  $M$  is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, V)$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of  $V$ -valued  $C^\infty$  forms on  $S$  (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

$$(3.1) \quad \begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma \backslash G \\ \kappa \downarrow & & \downarrow \lambda \\ M & \xrightarrow{\pi} & S \end{array}$$

Let  $\eta$  be an element of  $\mathcal{A}^n(S, V)$ , the space of global  $C^\infty$   $n$ -forms on  $M$  with values in  $V$ . Then

$$\phi = \kappa^* \pi^* \eta$$

is a  $V$ -valued form on  $G$  satisfying the equations

$$(3.2) \quad \begin{array}{ll} \text{i) } \gamma^* \phi = \rho(\gamma) \phi & \text{if } \gamma \in \Gamma \\ \text{ii) } \mathcal{L}_Y \phi = 0 & \text{if } Y \in \mathfrak{k}, \\ & \mathcal{L}_Y = \text{Lie derivative} = (\Lambda^n \text{Ad}^*)(Y) \\ \text{iii) } \iota_Y \phi = 0 & \text{if } Y \in \mathfrak{k} \\ & \iota_Y = \text{interior multiplication by } Y \end{array}$$

Conversely, every element  $\phi \in \mathcal{A}^n(G) \otimes_{\mathbb{C}} V$  ( $\mathcal{A}^n(G)$  denoting the space of  $C^\infty$   $n$ -forms on  $G$ ) that satisfies (3.2) is  $\kappa^* \pi^* \eta$  for some  $\eta \in \mathcal{A}^n(S, V)$ . We then apply the mapping  $\tilde{\Xi}$  of (2.6) to  $\phi$ , obtaining the  $n$ -form

$$(3.3) \quad \tilde{\eta} = \rho(g^{-1}) \phi$$

which satisfies

$$(3.4) \quad \begin{array}{ll} \text{i) } \gamma^* \tilde{\eta} = \tilde{\eta} & \text{if } \gamma \in \Gamma, \\ \text{ii) } \mathcal{L}_Y \tilde{\eta} = -\rho(Y) \tilde{\eta} & \text{if } Y \in \mathfrak{k}, \\ \text{iii) } \iota_Y \tilde{\eta} = 0 & \text{if } Y \in \mathfrak{k}. \end{array}$$

In particular, we may view  $\tilde{\eta}$  as a vector-valued form on  $\Gamma \backslash G$ .

We next describe the Hodge theory for  $H^n(S, \mathbf{V})$  from this point of view, as was done in [7] and [8]. Actually, one must work with the  $L_2$  cohomology when  $S$  is non-compact. Since we have defined a metric on  $A(\Gamma, \rho)$  in Section 2, and on the tangent bundle by the Killing form, there is an  $L_2$  norm  $\|\eta\|_{(2)}$  for  $\eta \in \mathcal{A}^n(S, \mathbf{V})$ , and the  $L_2$  cohomology is defined by

$$(3.5) \quad H_{(2)}^n(S, \mathbf{V}) = \frac{\{\eta \in \mathcal{A}^n(S, \mathbf{V}): \eta \text{ is } L_2 \text{ and } d\eta = 0\}}{\{\eta \text{ as above: } \eta = d\psi \text{ for some } L_2 \psi \in \mathcal{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

$$(3.6) \quad H_{(2)}^n(S, \mathbf{V}) \rightarrow H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) *Remark.* We may compute the  $L_2$  cohomology groups (3.5) from the complex of weakly differentiable  $L_2$  forms  $\mathcal{L}_{(2)}^*(S, \mathbf{V})$ ; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then  $d$  becomes a densely-defined differential for the “complex” of Hilbert spaces of  $\mathbf{V}$ -valued  $L_2$  forms, and

$$H_{(2)}^n(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed } \mathbf{V}\text{-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_2(n-1)\text{-forms}\}}.$$

We define the *reduced*  $L_2$  cohomology  $\bar{H}_{(2)}^n(S, \mathbf{V})$  by replacing the range of  $d$  in the above quotient by its Hilbert space closure; the reduced  $L_2$  cohomology inherits a Hilbert space structure from the  $L_2$  inner product.

In discussing  $\|\eta\|_{(2)}$ , we wish to make use of the form  $\tilde{\eta}$  of (3.4), and we have

(3.8) **LEMMA** [7, p. 380]. If  $\eta \in \mathcal{A}^n(S, \mathbf{V})$  and  $\tilde{\eta} \in \mathcal{A}^n(\Gamma \backslash G) \otimes V$  is the corresponding element, then

$$\|\eta\|_{(2)}^2 = c \|\tilde{\eta}\|_{(2)}^2,$$

where  $c$  equals the volume of  $K$ .

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis  $\{X_i\}_{i=1}^k$  of  $\mathfrak{p}^+$ , so

$$\{X_1, \bar{X}_1, \dots, X_k, \bar{X}_k\}$$

forms an orthonormal basis of  $\mathfrak{p}_{\mathbb{C}}$ . For  $\eta \in \mathcal{A}^{p,q}(S, V)$ , put

$$\eta_{i_1, \dots, i_p; j_1, \dots, j_q} = \tilde{\eta}(X_{i_1}, \dots, X_{i_p}, \bar{X}_{j_1}, \dots, \bar{X}_{j_q}) \in \mathcal{A}^0(G) \otimes V.$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative  $d$  on  $\mathcal{A}^\bullet(S, V)$  into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators  $D'$  and  $d'_p$  are defined by the formulas

$$\begin{aligned} (3.9) \quad & (D'\eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \dots, \widehat{i_u}, \dots, i_{p+1}; j_1, \dots, j_q}, \end{aligned}$$

$$\begin{aligned} (3.10) \quad & (d'_p \eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_u}) \eta_{i_1, \dots, \widehat{i_u}, \dots, i_{p+1}; j_1, \dots, j_q}. \end{aligned}$$

One also puts  $D'' = \overline{D'}$  and  $d''_p = \overline{d'_p}$ . Then  $d' = D' + d'_p$  and  $d'' = D'' + d''_p$ ; if we put  $D = D' + D''$  and  $d_p = d'_p + d''_p$ , then  $d = D + d_p$ . We remark that  $D$  gives a metric connection on  $\Phi(\rho)$ ; heuristically, we regard  $\kappa^*E(\rho)$  as being canonically flat.

Let  $\mathfrak{D}$  represent any of the above operators. One can obtain directly formulas for the  $L_2$  adjoint  $\mathfrak{D}^*$  and the Laplacian

$$(3.11) \quad \square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) PROPOSITION. As operators on  $\mathcal{A}^*(S, V)$ ,

- i)  $\square_d = \square_{d'} + \square_{d''}$
- ii)  $\square_d = \square_D + \square_{d_p}$
- iii)  $\square_D = \square_{D'} + \square_{D''}$
- iv)  $\square_{d_p} = \square_{d'_p} + \square_{d''_p}$
- v)  $\square_{d'} = \square_{D'} + \square_{d'_p}$

(3.13) Remark. One always has

$$\square_{(\mathfrak{D}_1 + \mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1 \mathfrak{D}_2^* + \mathfrak{D}_2^* \mathfrak{D}_1 + \mathfrak{D}_1^* \mathfrak{D}_2 + \mathfrak{D}_2 \mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since  $S$  is complete in the induced metric from  $M$ , the operators  $\mathfrak{D}$  as above have unique [3] closed extensions to  $\mathcal{L}_{(2)}^*(S, V)$ , so the identities (3.12) continue to remain valid in the strict sense on  $L_2$ . From this, one may conclude

(3.14) PROPOSITION. If  $\eta \in \mathcal{L}_{(2)}^*(S, V)$ , the following are equivalent:

- i)  $\square_d \eta = 0$  ( $\eta$  is harmonic),
- ii)  $\square_{d'} \eta = \square_{d''} \eta = 0$
- iii)  $\square_{D'} \eta = \square_{D''} \eta = \square_{d'_p} \eta = \square_{d''_p} \eta = 0$ ,
- iv)  $D' \eta = (D')^* \eta = D'' \eta = (D'')^* \eta = d'_p \eta$   
 $= (d'_p)^* \eta = d''_p \eta = (d''_p)^* \eta = 0$ .

Since  $\square_{\mathfrak{D}}$  is elliptic for any of the operators  $\mathfrak{D}$  above, harmonic forms are necessarily  $C^\infty$ . Let  $\mathcal{H}_{(2)}^n(S, V)$  denote the space of  $L_2$  harmonic  $n$ -forms with values in  $V$ . We obtain by standard theory (see [15, §1]):

(3.15) PROPOSITION. For all  $n$ ,

- i)  $\bar{H}_{(2)}^n(S, V) \simeq \mathcal{H}_{(2)}^n(S, V)$ ,
- ii) The mapping  $\mathcal{H}_{(2)}^n(S, V) \rightarrow H_{(2)}^n(S, V)$  is injective, and is an isomorphism if and only if  $d$ , operating on  $\mathcal{L}_{(2)}^{n-1}(S, V)$ , has closed range.

(3.16) *Remark.* An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that  $H_{(2)}^n(S, V)$  is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators  $d'$  and  $d''$ . Therefore, a form is harmonic if and only if its  $(p, q)$  components are harmonic, so

$$(3.17) \quad \mathcal{H}_{(2)}^n(S, V) = \bigoplus_{p+q=n} \mathcal{H}_{(2)}^{p,q}(S, V).$$

Passing this through the isomorphism (3.15, i), we get

$$(3.18) \quad \bar{H}_{(2)}^n(S, V) = \bigoplus_{p+q=n} H_{(2)}^{p,q}(S, V).$$

If we take  $S$  to be compact, we have  $H_{(2)}^n(S, V) = H^n(S, V)$ , and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) PROPOSITION [8, p. 14].

$$\square_{D''} + \square_{d'_p} = \square_{D'} + \square_{d''_p}.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY.  $\eta$  is harmonic if and only if

$$\square_{D''}\eta = \square_{d'_p}\eta = 0.$$

We close this section with a brief account of another way of viewing the cohomology groups  $H^n(\Gamma; \rho, V)$ , currently preferred in representation theory. For simplicity, we assume that  $S$  is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of  $\mathcal{A}^n(S, V)$  as a mapping from  $\Lambda^n \mathfrak{p}_C$  into  $\mathcal{A}^0(\Gamma \backslash G) \otimes V$  that satisfies a transformation rule under  $\mathfrak{k}$ . This correspondence gives an isomorphism of  $H^n(S, V)$  with the *relative Lie algebra cohomology* (see, e.g. [8, pp. 6-8] or [14, Ch. I]):

$$(3.21) \quad H^n(\mathfrak{g}_C, \mathfrak{k}_C, \mathcal{A}^0(\Gamma \backslash G) \otimes V),$$

associated to the cochain complex

$$(3.22) \quad \text{Hom}_K (\Lambda^* \mathfrak{p} \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

Here,  $\mathfrak{g}_\mathbb{C}$  acts on  $\mathcal{A}^0 (\Gamma \backslash G)$  by differentiation, induced by the regular representation of  $G$ .

(3.23) *Remark.* By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H_d^n (G, \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

For this reason, (3.21) is often referred to as "continuous cohomology."

The cohomology (3.21) decomposes according to the splitting of  $\mathcal{A}^0 (\Gamma \backslash G) \otimes V$ . First, one decomposes  $L_2 (\Gamma \backslash G)$  as a representation of  $G$ :

$$(3.24) \quad L_2 (\Gamma \backslash G) \simeq \widehat{\bigoplus}_{\alpha} E_{\alpha}$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

$$(3.25) \quad L_2 (\Gamma \backslash G, V) \simeq \widehat{\bigoplus}_{\alpha} (E_{\alpha} \otimes V)$$

Taking  $C^\infty$  vectors gives the decomposition

$$(3.26) \quad \mathcal{A}^0 (\Gamma \backslash G) \otimes V \simeq \widehat{\bigoplus}_{\alpha} (E_{\alpha}^{\infty} \otimes V),$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form  $\tilde{\eta}$ , the Laplacian is given by

$$(3.27) \quad \widetilde{\square \eta} = [-C + \rho(C)] \tilde{\eta},$$

where  $C$  is the Casimir element of the enveloping algebra of  $\mathfrak{g}$ . It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters  $\chi_{\alpha}$  of  $(\pi_{\alpha}, E_{\alpha})$  and  $\chi_{\rho}$  of  $(\rho, V)$  agree on  $C$ . In fact, if the space of harmonic forms is non-zero one must have  $\chi_{\alpha} = \chi_{\rho}$  (see [1, (2.4)]). In this case, every cochain with values in  $E_{\alpha}$  is harmonic. Thus,

$$(3.28) \quad \begin{aligned} H^n (S, V) &\simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} \text{Hom}_K (\Lambda^n \mathfrak{p}_{\mathbb{C}}, E_{\alpha} \otimes V) \\ &\simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} (\Lambda^n \mathfrak{p}_{\mathbb{C}}^* \otimes E_{\alpha} \otimes V)^K \quad (K\text{-invariants}). \end{aligned}$$

From (3.27) and (3.28), one obtains the following:

(3.29) PROPOSITION. Let  $(\rho_1, V_1)$  and  $(\rho_2, V_2)$  be two irreducible representations of  $G$ , and suppose that  $\rho_1(C) = \rho_2(C)$ . Then every morphism of  $K$ -representations

$$\phi: \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \rightarrow \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_*: \mathcal{H}^{n_1}(S, V_1) \rightarrow \mathcal{H}^{n_2}(S, V_2).$$

and thus a mapping  $\phi_*: H^{n_1}(S, V_1) \rightarrow H^{n_2}(S, V_2)$ . (If the infinitesimal characters of  $(\rho_1, V_1)$  and  $(\rho_2, V_2)$  differ, then  $\phi_*$  is the zero mapping.)

If we now decompose each  $\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V$  as a representation of  $K$  and apply (3.29) to the projections onto each component, there is induced decomposition of  $H^n(S, V)$ , much in the spirit of [2]. If we decompose only  $\Lambda^n \mathfrak{p}^*$ , we obtain the decomposition (3.18). We will refine that decomposition in §5.

If  $S$  is non-compact, then  $L_2(\Gamma \backslash G)$  is the direct sum of its discrete spectrum  $L_2(\Gamma \backslash G)_d$  and the continuous spectrum  $L_2(\Gamma \backslash G)_{ct}$ . One then has a decomposition like (3.24) only for  $L_2(\Gamma \backslash G)_d$ . From there, one obtains an injection

$$(3.30) \quad \bigoplus_{\alpha} (E_{\alpha}^{\infty} \otimes V) \rightarrow \mathcal{A}_{(2)}^0(\Gamma \backslash G) \otimes V,$$

whose image consists of those  $C^\infty$   $V$ -valued functions for which all left-invariant differential operators are in  $L_2$ . Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if  $\Gamma$  is an arithmetic subgroup of  $G$ , then all harmonic forms come from  $L_2(\Gamma \backslash G)_d$ . In this case, one therefore obtains, as in (3.28), the isomorphism

$$(3.31) \quad \bar{H}_{(2)}^n(S, V) \simeq \bigoplus_{\chi_\alpha = \chi_p} (\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V)^K.$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced  $L_2$  cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced)  $L_2$  cohomology is for some groups infinite-dimensional, with  $d$  having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein <sup>1)</sup>. (See also [12] for a different approach to the  $L_2$  cohomology.)

<sup>1)</sup> See note added in proof.