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§3. The cohomology groups $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups $H^n(\Gamma; \rho, V)$ for a finite-dimensional representation (ρ, V) of G, which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since M is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, \mathbf{V})$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of $\mathbf{V}$-valued C^∞ forms on S (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

$$\text{(3.1)} \qquad \begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma\backslash G \\ \kappa \downarrow & & \downarrow \lambda \\ M & \xrightarrow{\pi} & S \end{array}$$

Let η be an element of $\mathscr{A}^n(S, \mathbf{V})$, the space of global C^∞ n-forms on M with values in $\mathbf{V}$. Then

$$\phi = \kappa^*\pi^*\eta$$

is a V-valued form on G satisfying the equations

(3.2)
i) $\gamma^*\phi = \rho(\gamma)\phi$ if $\gamma \in \Gamma$

ii) $\mathscr{L}_Y\phi = 0$ if $Y \in \mathfrak{k}$,

$\mathscr{L}_Y =$ Lie derivative $= (\Lambda^n \mathrm{Ad}^*)(Y)$

iii) $\iota_Y\phi = 0$ if $Y \in \mathfrak{k}$

$\iota_Y =$ interior multiplication by Y

Conversely, every element $\phi \in \mathscr{A}^n(G) \otimes_{\mathbf{C}} V$ ($\mathscr{A}^n(G)$ denoting the space of C^∞ n-forms on G) that satisfies (3.2) is $\kappa^*\pi^*\eta$ for some $\eta \in \mathscr{A}^n(S, \mathbf{V})$. We then apply the mapping $\tilde{\Xi}$ of (2.6) to ϕ, obtaining the n-form

$$\text{(3.3)} \qquad \tilde{\eta} = \rho(g^{-1})\phi$$

which satisfies

$$\text{(3.4)} \qquad \begin{aligned} &\text{i)}\ \gamma^*\tilde{\eta} = \tilde{\eta} && \text{if} \quad \gamma \in \Gamma, \\ &\text{ii)}\ \mathscr{L}_Y\tilde{\eta} = -\rho(Y)\,\tilde{\eta} && \text{if} \quad Y \in \mathfrak{k}, \\ &\text{iii)}\ \iota_Y\tilde{\eta} = 0 && \text{if} \quad Y \in \mathfrak{k}. \end{aligned}$$

In particular, we may view $\tilde{\eta}$ as a vector-valued form on $\Gamma\backslash G$.

We next describe the Hodge theory for $H^n(S, \mathbf{V})$ from this point of view, as was done in [7] and [8]. Actually, one must work with the L_2 cohomology when S is non-compact. Since we have defined a metric on $A(\Gamma, \rho)$ in Section 2, and on the tangent bundle by the Killing form, there is an L_2 norm $\| \eta \|_{(2)}$ for $\eta \in \mathscr{A}^n(S, \mathbf{V})$, and the L_2 cohomology is defined by

(3.5)
$$H^n_{(2)}(S, \mathbf{V}) = \frac{\{\eta \in \mathscr{A}^n(S, \mathbf{V}) : \ \eta \text{ is } L_2 \text{ and } d\eta = 0\}}{\{\eta \text{ as above}: \ \eta = d\psi \text{ for some } L_2 \ \psi \in \mathscr{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

$$\text{(3.6)} \qquad H^n_{(2)}(S, \mathbf{V}) \to H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) *Remark.* We may compute the L_2 cohomology groups (3.5) from the complex of weakly differentiable L_2 forms $\mathscr{L}^{\bullet}_{(2)}(S, \mathbf{V})$; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then d becomes a densely-defined differential for the "complex" of Hilbert spaces of $\mathbf{V}$-valued L_2 forms, and

$$H^n_{(2)}(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed } \mathbf{V}\text{-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_2\ (n-1)\text{-forms}\}}.$$

We define the *reduced* L_2 cohomology $\bar{H}^n_{(2)}(S, \mathbf{V})$ by replacing the range of d in the above quotient by its Hilbert space closure; the reduced L_2 cohomology inherits a Hilbert space structure from the L_2 inner product.

In discussing $\| \eta \|_{(2)}$, we wish to make use of the form $\tilde{\eta}$ of (3.4), and we have

(3.8) Lemma [7, p. 380]. If $\eta \in \mathscr{A}^n(S, \mathbf{V})$ and $\tilde{\eta} \in \mathscr{A}^n(\Gamma\backslash G) \otimes V$ is the corresponding element, then

$$\| \eta \|^2_{(2)} = c \, \| \tilde{\eta} \|^2_{(2)},$$

where c equals the volume of K.

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis $\{X_i\}_{i=1}^k$ of $\mathfrak{p}^+$, so

$$\{X_1, \bar{X}_1, \ldots, X_k, \bar{X}_k\}$$

forms an orthonormal basis of $\mathfrak{p}_{\mathbf{C}}$. For $\eta \in \mathscr{A}^{p,q}(S, \mathbf{V})$, put

$$\eta_{i_1, \ldots, i_p; j_1, \ldots, j_q} = \tilde{\eta}(X_{i_1}, \ldots, X_{i_p}, \bar{X}_{j_1}, \ldots, \bar{X}_{j_q}) \in \mathscr{A}^0(G) \otimes V .$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative d on $\mathscr{A}^\bullet(S, \mathbf{V})$ into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators D' and d'_p are defined by the formulas

(3.9)
$$(D'\eta)_{i_1, \ldots, i_{p+1}; j_1, \ldots, j_q} = \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \ldots, \hat{i_u}, \ldots, i_{p+1}; j_1, \ldots, j_q},$$

(3.10)
$$(d'_\rho\eta)_{i_1, \ldots, i_{p+1}; j_1, \ldots, j_q} = \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_u}) \eta_{i_1, \ldots, \hat{i_u}, \ldots, i_{p+1}; j_1, \ldots, j_q}.$$

One also puts $D'' = \overline{D'}$ and $d''_\rho = \overline{d'_\rho}$. Then $d' = D' + d'_\rho$ and $d'' = D'' + d''_\rho$; if we put $D = D' + D''$ and $d_\rho = d'_\rho + d''_\rho$, then $d = D + d_\rho$. We remark that D gives a metric connection on $\Phi(\rho)$; heuristically, we regard $\kappa^* E(\rho)$ as being canonically flat.

Let $\mathfrak{D}$ represent any of the above operators. One can obtain directly formulas for the L_2 adjoint $\mathfrak{D}^*$ and the Laplacian

(3.11)
$$\square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) PROPOSITION. *As operators on* $\mathscr{A}^\bullet(S, \mathbf{V})$,

i) $\square_d = \square_{d'} + \square_{d''}$

ii) $\square_d = \square_D + \square_{d_\rho}$

iii) $\square_D = \square_{D'} + \square_{D''}$

iv) $\square_{d_\rho} = \square_{d'_\rho} + \square_{d''_\rho}$

v) $\square_{d'} = \square_{D'} + \square_{d'_\rho}$

(3.13) *Remark.* One always has

$$\square_{(\mathfrak{D}_1+\mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1\mathfrak{D}_2^* + \mathfrak{D}_2^*\mathfrak{D}_1 + \mathfrak{D}_1^*\mathfrak{D}_2 + \mathfrak{D}_2\mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since S is complete in the induced metric from M, the operators $\mathfrak{D}$ as above have unique [3] closed extensions to $\mathscr{L}^\bullet_{(2)}(S, \mathbf{V})$, so the identities (3.12) continue to remain valid in the strict sense on L_2. From this, one may conclude

(3.14) PROPOSITION. *If* $\eta \in \mathscr{L}^\bullet_{(2)}(S, \mathbf{V})$, *the following are equivalent:*

i) $\square_d\eta = 0$ (η *is harmonic*),

ii) $\square_{d'}\eta = \square_{d''}\eta = 0$

iii) $\square_{D'}\eta = \square_{D''}\eta = \square_{d'_\rho}\eta = \square_{d''_\rho}\eta = 0,$

iv) $D'\eta = (D')^*\eta = D''\eta = (D'')^*\eta = d'_\rho\eta$
$= (d'_\rho)^*\eta = d''_\rho\eta = (d''_\rho)^*\eta = 0\,.$

Since $\square_{\mathfrak{D}}$ is elliptic for any of the operators $\mathfrak{D}$ above, harmonic forms are necessarily C^∞. Let $\mathscr{h}^n_{(2)}(S, \mathbf{V})$ denote the space of L_2 harmonic n-forms with values in $\mathbf{V}$. We obtain by standard theory (see [15, §1]):

(3.15) PROPOSITION. *For all* n,

i) $\bar{H}^n_{(2)}(S, \mathbf{V}) \simeq \mathscr{h}^n_{(2)}(S, \mathbf{V})$,

ii) *The mapping* $\mathscr{h}^n_{(2)}(S, \mathbf{V}) \to H^n_{(2)}(S, \mathbf{V})$ *is injective, and is an isomorphism if and only if* d, *operating on* $\mathscr{L}^{n-1}_{(2)}(S, \mathbf{V})$, *has closed range.*

(3.16) *Remark.* An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that $H^n_{(2)}(S, \mathbf{V})$ is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators d' and d''. Therefore, a form is harmonic if and only if its (p, q) components are harmonic, so

$$(3.17) \qquad \mathscr{h}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{p+q=n} \mathscr{h}^{p,q}_{(2)}(S, \mathbf{V}).$$

Passing this through the isomorphism (3.15, i), we get

$$(3.18) \qquad \bar{H}^n_{(2)}(S, \mathbf{V}) = \bigoplus_{p+q=n} H^{p,q}_{(2)}(S, \mathbf{V}).$$

If we take S to be compact, we have $H^n_{(2)}(S, \mathbf{V}) = H^n(S, \mathbf{V})$, and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) PROPOSITION [8, p. 14].

$$\Box_{D''} + \Box_{d'_\rho} = \Box_{D'} + \Box_{d''_\rho}.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY. η is harmonic if and only if

$$\Box_{D''}\eta = \Box_{d'_\rho}\eta = 0.$$

We close this section with a brief account of another way of viewing the cohomology groups $H^n(\Gamma; \rho, V)$, currently preferred in representation theory. For simplicity, we assume that S is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of $\mathscr{A}^n(S, \mathbf{V})$ as a mapping from $\Lambda^n \mathfrak{p}_{\mathbf{C}}$ into $\mathscr{A}^0(\Gamma\backslash G) \otimes V$ that satisfies a transformation rule under $\mathfrak{k}$. This correspondence gives an isomorphism of $H^n(S, \mathbf{V})$ with the *relative Lie algebra cohomology* (see, e.g. [8, pp. 6-8] or [14, Ch. I]):

$$(3.21) \qquad H^n(\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}}, \mathscr{A}^0(\Gamma\backslash G) \otimes V),$$

associated to the cochain complex

$$\mathrm{Hom}_K\,(\Lambda^{\bullet}\mathfrak{p}\,\mathscr{A}^0\,(\Gamma\backslash G)\otimes V)\,. \tag{3.22}$$

Here, $\mathfrak{g}_{\mathbf{C}}$ acts on $\mathscr{A}^0\,(\Gamma\backslash G)$ by differentiation, induced by the regular representation of G.

(3.23) *Remark.* By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H^n_d\,(G, \mathscr{A}^0\,(\Gamma\backslash G)\otimes V)\,.$$

For this reason, (3.21) is often referred to as "continuous cohomology."

The cohomology (3.21) decomposes according to the splitting of $\mathscr{A}^0\,(\Gamma\backslash G)\otimes V$. First, one decomposes $L_2\,(\Gamma\backslash G)$ as a representation of G:

$$L_2\,(\Gamma\backslash G)\simeq \widehat{\bigoplus_{\alpha}}\, E_\alpha \tag{3.24}$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

$$L_2\,(\Gamma\backslash G, V)\simeq \widehat{\bigoplus_{\alpha}}\,(E_\alpha\otimes V) \tag{3.25}$$

Taking C^∞ vectors gives the decomposition

$$\mathscr{A}^0\,(\Gamma\backslash G)\otimes V\simeq \widehat{\bigoplus_{\alpha}}\,(E^\infty_\alpha\otimes V)\,, \tag{3.26}$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form $\tilde{\eta}$, the Laplacian is given by

$$\widetilde{\Box\eta} = [-C + \rho\,(C)]\,\tilde{\eta}\,, \tag{3.27}$$

where C is the Casimir element of the enveloping algebra of $\mathfrak{g}$. It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters χ_α of (π_α, E_α) and χ_ρ of (ρ, V) agree on C. In fact, if the space of harmonic forms is non-zero one must have $\chi_\alpha = \chi_\rho$ (see [1, (2.4)]). In this case, every cochain with values in E_α is harmonic. Thus,

$$\begin{aligned} H^n\,(S, \mathbf{V}) &\simeq \bigoplus_{\chi_\alpha=\chi_\rho} \mathrm{Hom}_K\,(\Lambda^n\mathfrak{p}_{\mathbf{C}}, E_\alpha\otimes V) \\ &\simeq \bigoplus_{\chi_\alpha=\chi_\rho} (\Lambda^n\mathfrak{p}^*_{\mathbf{C}}\otimes E_\alpha\otimes V)^K \quad (K\text{-invariants}). \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), one obtains the following:

(3.29) PROPOSITION. *Let* (ρ_1, V_1) *and* (ρ_2, V_2) *be two irreducible representations of* G, *and suppose that* $\rho_1(C) = \rho_2(C)$. *Then every morphism of* K-*representations*

$$\phi : \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \to \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_* : \mathscr{H}^{n_1}(S, \mathbf{V}_1) \to \mathscr{H}^{n_2}(S, \mathbf{V}_2).$$

and thus a mapping $\phi_* : H^{n_1}(S, \mathbf{V}_1) \to H^{n_2}(S, \mathbf{V}_2)$. *(If the infinitesimal characters of* (ρ_1, V_1) *and* (ρ_2, V_2) *differ, then* ϕ_* *is the zero mapping.)*

If we now decompose each $\Lambda^n \mathfrak{p}_{\mathbf{C}}^* \otimes E_\alpha \otimes V$ as a representation of K and apply (3.29) to the projections onto each component, there is induced decomposition of $H^n(S, \mathbf{V})$, much in the spirit of [2]. If we decompose only $\Lambda^n \mathfrak{p}^*$, we obtain the decomposition (3.18). We will refine that decomposition in §5.

If S is non-compact, then $L_2(\Gamma \backslash G)$ is the direct sum of its discrete spectrum $L_2(\Gamma \backslash G)_d$ and the continuous spectrum $L_2(\Gamma \backslash G)_{ct}$. One then has a decomposition like (3.24) only for $L_2(\Gamma \backslash G)_d$. From there, one obtains an injection

$$\widehat{\bigoplus_\alpha}' (E_\alpha^\infty \otimes V) \to \mathscr{A}_{(2)}^0(\Gamma \backslash G) \otimes V, \tag{3.30}$$

whose image consists of those C^∞ $\mathbf{V}$-valued functions for which all left-invariant differential operators are in L_2. Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if Γ is an arithmetic subgroup of G, then all harmonic forms come from $L_2(\Gamma \backslash G)_d$. In this case, one therefore obtains, as in (3.28), the isomorphism

$$\bar{H}_{(2)}^n(S, \mathbf{V}) \simeq \bigoplus_{\chi_\alpha = \chi_\rho} (\Lambda^n \mathfrak{p}_{\mathbf{C}}^* \otimes E_\alpha \otimes V)^K. \tag{3.31}$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced L_2 cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced) L_2 cohomology is for some groups infinite-dimensional, with d having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein [1]). (See also [12] for a different approach to the L_2 cohomology.)

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[1]) See note added in proof.