

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 27 (1981)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE  
**Autor:** Zucker, Steven  
**Kapitel:** §3. The cohomology groups  $H^n(\Gamma; \rho, V)$   
**DOI:** <https://doi.org/10.5169/seals-51751>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 14.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

§3. THE COHOMOLOGY GROUPS  $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups  $H^n(\Gamma; \rho, V)$  for a finite-dimensional representation  $(\rho, V)$  of  $G$ , which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since  $M$  is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, V)$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of  $V$ -valued  $C^\infty$  forms on  $S$  (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

$$(3.1) \quad \begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma \backslash G \\ \kappa \downarrow & & \downarrow \lambda \\ M & \xrightarrow{\pi} & S \end{array}$$

Let  $\eta$  be an element of  $\mathcal{A}^n(S, V)$ , the space of global  $C^\infty$   $n$ -forms on  $M$  with values in  $V$ . Then

$$\phi = \kappa^* \pi^* \eta$$

is a  $V$ -valued form on  $G$  satisfying the equations

$$(3.2) \quad \begin{array}{ll} \text{i) } \gamma^* \phi = \rho(\gamma) \phi & \text{if } \gamma \in \Gamma \\ \text{ii) } \mathcal{L}_Y \phi = 0 & \text{if } Y \in \mathfrak{k} \\ & \mathcal{L}_Y = \text{Lie derivative} = (\Lambda^n \text{Ad}^*)(Y) \\ \text{iii) } \iota_Y \phi = 0 & \text{if } Y \in \mathfrak{k} \\ & \iota_Y = \text{interior multiplication by } Y \end{array}$$

Conversely, every element  $\phi \in \mathcal{A}^n(G) \otimes_{\mathbb{C}} V$  ( $\mathcal{A}^n(G)$  denoting the space of  $C^\infty$   $n$ -forms on  $G$ ) that satisfies (3.2) is  $\kappa^* \pi^* \eta$  for some  $\eta \in \mathcal{A}^n(S, V)$ . We then apply the mapping  $\tilde{\Xi}$  of (2.6) to  $\phi$ , obtaining the  $n$ -form

$$(3.3) \quad \tilde{\eta} = \rho(g^{-1}) \phi$$

which satisfies

$$(3.4) \quad \begin{array}{ll} \text{i) } \gamma^* \tilde{\eta} = \tilde{\eta} & \text{if } \gamma \in \Gamma, \\ \text{ii) } \mathcal{L}_Y \tilde{\eta} = -\rho(Y) \tilde{\eta} & \text{if } Y \in \mathfrak{k}, \\ \text{iii) } \iota_Y \tilde{\eta} = 0 & \text{if } Y \in \mathfrak{k}. \end{array}$$

In particular, we may view  $\tilde{\eta}$  as a vector-valued form on  $\Gamma \backslash G$ .

We next describe the Hodge theory for  $H^n(S, \mathbf{V})$  from this point of view, as was done in [7] and [8]. Actually, one must work with the  $L_2$  cohomology when  $S$  is non-compact. Since we have defined a metric on  $A(\Gamma, \rho)$  in Section 2, and on the tangent bundle by the Killing form, there is an  $L_2$  norm  $\|\eta\|_{(2)}$  for  $\eta \in \mathcal{A}^n(S, \mathbf{V})$ , and the  $L_2$  cohomology is defined by

$$(3.5) \quad H_{(2)}^n(S, \mathbf{V}) = \frac{\{\eta \in \mathcal{A}^n(S, \mathbf{V}) : \eta \text{ is } L_2 \text{ and } d\eta = 0\}}{\{\eta \text{ as above: } \eta = d\psi \text{ for some } L_2 \psi \in \mathcal{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

$$(3.6) \quad H_{(2)}^n(S, \mathbf{V}) \rightarrow H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) *Remark.* We may compute the  $L_2$  cohomology groups (3.5) from the complex of weakly differentiable  $L_2$  forms  $\mathcal{L}_{(2)}^*(S, \mathbf{V})$ ; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then  $d$  becomes a densely-defined differential for the “complex” of Hilbert spaces of  $\mathbf{V}$ -valued  $L_2$  forms, and

$$H_{(2)}^n(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed } \mathbf{V}\text{-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_2(n-1)\text{-forms}\}}.$$

We define the *reduced*  $L_2$  cohomology  $\bar{H}_{(2)}^n(S, \mathbf{V})$  by replacing the range of  $d$  in the above quotient by its Hilbert space closure; the reduced  $L_2$  cohomology inherits a Hilbert space structure from the  $L_2$  inner product.

In discussing  $\|\eta\|_{(2)}$ , we wish to make use of the form  $\tilde{\eta}$  of (3.4), and we have

(3.8) LEMMA [7, p. 380]. If  $\eta \in \mathcal{A}^n(S, \mathbf{V})$  and  $\tilde{\eta} \in \mathcal{A}^n(\Gamma \backslash G) \otimes V$  is the corresponding element, then

$$\|\eta\|_{(2)}^2 = c \|\tilde{\eta}\|_{(2)}^2,$$

where  $c$  equals the volume of  $K$ .

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis  $\{X_i\}_{i=1}^k$  of  $\mathfrak{p}^+$ , so

$$\{X_1, \bar{X}_1, \dots, X_k, \bar{X}_k\}$$

forms an orthonormal basis of  $\mathfrak{p}_{\mathbb{C}}$ . For  $\eta \in \mathcal{A}^{p,q}(S, V)$ , put

$$\eta_{i_1, \dots, i_p; j_1, \dots, j_q} = \tilde{\eta}(X_{i_1}, \dots, X_{i_p}, \bar{X}_{j_1}, \dots, \bar{X}_{j_q}) \in \mathcal{A}^0(G) \otimes V.$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative  $d$  on  $\mathcal{A}^*(S, V)$  into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators  $D'$  and  $d'_p$  are defined by the formulas

$$(3.9) \quad \begin{aligned} & (D'\eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \dots, \hat{i}_u, \dots, i_{p+1}; j_1, \dots, j_q}, \end{aligned}$$

$$(3.10) \quad \begin{aligned} & (d'_p \eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_u}) \eta_{i_1, \dots, \hat{i}_u, \dots, i_{p+1}; j_1, \dots, j_q}. \end{aligned}$$

One also puts  $D'' = \overline{D'}$  and  $d''_p = \overline{d'_p}$ . Then  $d' = D' + d'_p$  and  $d'' = D'' + d''_p$ ; if we put  $D = D' + D''$  and  $d_p = d'_p + d''_p$ , then  $d = D + d_p$ . We remark that  $D$  gives a metric connection on  $\Phi(\rho)$ ; heuristically, we regard  $\kappa^*E(\rho)$  as being canonically flat.

Let  $\mathfrak{D}$  represent any of the above operators. One can obtain directly formulas for the  $L_2$  adjoint  $\mathfrak{D}^*$  and the Laplacian

$$(3.11) \quad \square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) PROPOSITION. As operators on  $\mathcal{A}^\bullet(S, \mathbf{V})$ ,

- i)  $\square_d = \square_{d'} + \square_{d''}$
- ii)  $\square_d = \square_D + \square_{d_p}$
- iii)  $\square_D = \square_{D'} + \square_{D''}$
- iv)  $\square_{d_p} = \square_{d'_p} + \square_{d''_p}$
- v)  $\square_{d'} = \square_{D'} + \square_{d'_p}$

(3.13) Remark. One always has

$$\square_{(\mathfrak{D}_1 + \mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1 \mathfrak{D}_2^* + \mathfrak{D}_2^* \mathfrak{D}_1 + \mathfrak{D}_1^* \mathfrak{D}_2 + \mathfrak{D}_2 \mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since  $S$  is complete in the induced metric from  $M$ , the operators  $\mathfrak{D}$  as above have unique [3] closed extensions to  $\mathcal{L}^\bullet_{(2)}(S, \mathbf{V})$ , so the identities (3.12) continue to remain valid in the strict sense on  $L_2$ . From this, one may conclude

(3.14) PROPOSITION. If  $\eta \in \mathcal{L}^\bullet_{(2)}(S, \mathbf{V})$ , the following are equivalent:

- i)  $\square_d \eta = 0$  ( $\eta$  is harmonic),
- ii)  $\square_{d'} \eta = \square_{d''} \eta = 0$
- iii)  $\square_{D'} \eta = \square_{D''} \eta = \square_{d'_p} \eta = \square_{d''_p} \eta = 0$ ,
- iv)  $D' \eta = (D')^* \eta = D'' \eta = (D'')^* \eta = d'_p \eta = (d'_p)^* \eta = d''_p \eta = (d''_p)^* \eta = 0$ .

Since  $\square_{\mathfrak{D}}$  is elliptic for any of the operators  $\mathfrak{D}$  above, harmonic forms are necessarily  $C^\infty$ . Let  $\mathcal{H}^n_{(2)}(S, \mathbf{V})$  denote the space of  $L_2$  harmonic  $n$ -forms with values in  $\mathbf{V}$ . We obtain by standard theory (see [15, §1]):

(3.15) PROPOSITION. For all  $n$ ,

- i)  $\bar{H}^n_{(2)}(S, \mathbf{V}) \simeq \mathcal{H}^n_{(2)}(S, \mathbf{V})$ ,
- ii) The mapping  $\mathcal{H}^n_{(2)}(S, \mathbf{V}) \rightarrow H^n_{(2)}(S, \mathbf{V})$  is injective, and is an isomorphism if and only if  $d$ , operating on  $\mathcal{L}^{n-1}_{(2)}(S, \mathbf{V})$ , has closed range.

(3.16) *Remark.* An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that  $H_{(2)}^n(S, \mathbf{V})$  is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators  $d'$  and  $d''$ . Therefore, a form is harmonic if and only if its  $(p, q)$  components are harmonic, so

$$(3.17) \quad \mathcal{H}_{(2)}^n(S, \mathbf{V}) = \bigoplus_{p+q=n} \mathcal{H}_{(2)}^{p,q}(S, \mathbf{V}).$$

Passing this through the isomorphism (3.15, i), we get

$$(3.18) \quad \bar{H}_{(2)}^n(S, \mathbf{V}) = \bigoplus_{p+q=n} H_{(2)}^{p,q}(S, \mathbf{V}).$$

If we take  $S$  to be compact, we have  $H_{(2)}^n(S, \mathbf{V}) = H^n(S, \mathbf{V})$ , and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) PROPOSITION [8, p. 14].

$$\square_{D''} + \square_{d'_p} = \square_{D'} + \square_{d''_p}.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY.  $\eta$  is harmonic if and only if

$$\square_{D''}\eta = \square_{d'_p}\eta = 0.$$

We close this section with a brief account of another way of viewing the cohomology groups  $H^n(\Gamma; \rho, V)$ , currently preferred in representation theory. For simplicity, we assume that  $S$  is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of  $\mathcal{A}^n(S, \mathbf{V})$  as a mapping from  $\Lambda^n \mathfrak{p}_{\mathbb{C}}$  into  $\mathcal{A}^0(\Gamma \backslash G) \otimes V$  that satisfies a transformation rule under  $\mathfrak{k}$ . This correspondence gives an isomorphism of  $H^n(S, \mathbf{V})$  with the *relative Lie algebra cohomology* (see, e.g. [8, pp. 6-8] or [14, Ch. I]):

$$(3.21) \quad H^n(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathcal{A}^0(\Gamma \backslash G) \otimes V),$$

associated to the cochain complex

$$(3.22) \quad \text{Hom}_K (\Lambda^p \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

Here,  $\mathfrak{g}_\mathbb{C}$  acts on  $\mathcal{A}^0 (\Gamma \backslash G)$  by differentiation, induced by the regular representation of  $G$ .

(3.23) *Remark.* By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H_d^n (G, \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

For this reason, (3.21) is often referred to as “continuous cohomology.”

The cohomology (3.21) decomposes according to the splitting of  $\mathcal{A}^0 (\Gamma \backslash G) \otimes V$ . First, one decomposes  $L_2 (\Gamma \backslash G)$  as a representation of  $G$ :

$$(3.24) \quad L_2 (\Gamma \backslash G) \simeq \widehat{\bigoplus}_\alpha E_\alpha$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

$$(3.25) \quad L_2 (\Gamma \backslash G, V) \simeq \widehat{\bigoplus}_\alpha (E_\alpha \otimes V)$$

Taking  $C^\infty$  vectors gives the decomposition

$$(3.26) \quad \mathcal{A}^0 (\Gamma \backslash G) \otimes V \simeq \widehat{\bigoplus}_\alpha (E_\alpha^\infty \otimes V),$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form  $\widetilde{\eta}$ , the Laplacian is given by

$$(3.27) \quad \widetilde{\square} \eta = [-C + \rho(C)] \widetilde{\eta},$$

where  $C$  is the Casimir element of the enveloping algebra of  $\mathfrak{g}$ . It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters  $\chi_\alpha$  of  $(\pi_\alpha, E_\alpha)$  and  $\chi_\rho$  of  $(\rho, V)$  agree on  $C$ . In fact, if the space of harmonic forms is non-zero one must have  $\chi_\alpha = \chi_\rho$  (see [1, (2.4)]). In this case, every cochain with values in  $E_\alpha$  is harmonic. Thus,

$$(3.28) \quad \begin{aligned} H^n (S, V) &\simeq \bigoplus_{\chi_\alpha = \chi_\rho} \text{Hom}_K (\Lambda^n \mathfrak{p}_\mathbb{C}, E_\alpha \otimes V) \\ &\simeq \bigoplus_{\chi_\alpha = \chi_\rho} (\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V)^K \quad (K\text{-invariants}). \end{aligned}$$

From (3.27) and (3.28), one obtains the following:

(3.29) PROPOSITION. Let  $(\rho_1, V_1)$  and  $(\rho_2, V_2)$  be two irreducible representations of  $G$ , and suppose that  $\rho_1(C) = \rho_2(C)$ . Then every morphism of  $K$ -representations

$$\phi: \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \rightarrow \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_*: \mathcal{H}^{n_1}(S, V_1) \rightarrow \mathcal{H}^{n_2}(S, V_2).$$

and thus a mapping  $\phi_*: H^{n_1}(S, V_1) \rightarrow H^{n_2}(S, V_2)$ . (If the infinitesimal characters of  $(\rho_1, V_1)$  and  $(\rho_2, V_2)$  differ, then  $\phi_*$  is the zero mapping.)

If we now decompose each  $\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V$  as a representation of  $K$  and apply (3.29) to the projections onto each component, there is induced decomposition of  $H^n(S, V)$ , much in the spirit of [2]. If we decompose only  $\Lambda^n \mathfrak{p}^*$ , we obtain the decomposition (3.18). We will refine that decomposition in §5.

If  $S$  is non-compact, then  $L_2(\Gamma \backslash G)$  is the direct sum of its discrete spectrum  $L_2(\Gamma \backslash G)_d$  and the continuous spectrum  $L_2(\Gamma \backslash G)_{ct}$ . One then has a decomposition like (3.24) only for  $L_2(\Gamma \backslash G)_d$ . From there, one obtains an injection

$$(3.30) \quad \widehat{\bigoplus_\alpha (E_\alpha^\infty \otimes V)} \rightarrow \mathcal{A}_{(2)}^0(\Gamma \backslash G) \otimes V,$$

whose image consists of those  $C^\infty V$ -valued functions for which all left-invariant differential operators are in  $L_2$ . Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if  $\Gamma$  is an arithmetic subgroup of  $G$ , then all harmonic forms come from  $L_2(\Gamma \backslash G)_d$ . In this case, one therefore obtains, as in (3.28), the isomorphism

$$(3.31) \quad \bar{H}_{(2)}^n(S, V) \simeq \bigoplus_{\chi_\alpha = \chi_p} (\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V)^K.$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced  $L_2$  cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced)  $L_2$  cohomology is for some groups infinite-dimensional, with  $d$  having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein <sup>1)</sup>. (See also [12] for a different approach to the  $L_2$  cohomology.)

<sup>1)</sup> See note added in proof.