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§3. THE COHOMOLOGY GROUPS $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups $H^n(\Gamma; \rho, V)$ for a finite-dimensional representation (ρ, V) of G , which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since M is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, V)$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of V -valued C^∞ forms on S (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

$$(3.1) \quad \begin{array}{ccc} G & \xrightarrow{\psi} & \Gamma \backslash G \\ \kappa \downarrow & & \downarrow \lambda \\ M & \xrightarrow{\pi} & S \end{array}$$

Let η be an element of $\mathcal{A}^n(S, V)$, the space of global C^∞ n -forms on M with values in V . Then

$$\phi = \kappa^* \pi^* \eta$$

is a V -valued form on G satisfying the equations

$$(3.2) \quad \begin{array}{ll} \text{i) } \gamma^* \phi = \rho(\gamma) \phi & \text{if } \gamma \in \Gamma \\ \text{ii) } \mathcal{L}_Y \phi = 0 & \text{if } Y \in \mathfrak{k}, \\ & \mathcal{L}_Y = \text{Lie derivative} = (\Lambda^n \text{Ad}^*)(Y) \\ \text{iii) } \iota_Y \phi = 0 & \text{if } Y \in \mathfrak{k} \\ & \iota_Y = \text{interior multiplication by } Y \end{array}$$

Conversely, every element $\phi \in \mathcal{A}^n(G) \otimes_{\mathbb{C}} V$ ($\mathcal{A}^n(G)$ denoting the space of C^∞ n -forms on G) that satisfies (3.2) is $\kappa^* \pi^* \eta$ for some $\eta \in \mathcal{A}^n(S, V)$. We then apply the mapping $\tilde{\Xi}$ of (2.6) to ϕ , obtaining the n -form

$$(3.3) \quad \tilde{\eta} = \rho(g^{-1}) \phi$$

which satisfies

$$(3.4) \quad \begin{array}{ll} \text{i) } \gamma^* \tilde{\eta} = \tilde{\eta} & \text{if } \gamma \in \Gamma, \\ \text{ii) } \mathcal{L}_Y \tilde{\eta} = -\rho(Y) \tilde{\eta} & \text{if } Y \in \mathfrak{k}, \\ \text{iii) } \iota_Y \tilde{\eta} = 0 & \text{if } Y \in \mathfrak{k}. \end{array}$$

In particular, we may view $\tilde{\eta}$ as a vector-valued form on $\Gamma \backslash G$.

We next describe the Hodge theory for $H^n(S, \mathbf{V})$ from this point of view, as was done in [7] and [8]. Actually, one must work with the L_2 cohomology when S is non-compact. Since we have defined a metric on $A(\Gamma, \rho)$ in Section 2, and on the tangent bundle by the Killing form, there is an L_2 norm $\|\eta\|_{(2)}$ for $\eta \in \mathcal{A}^n(S, \mathbf{V})$, and the L_2 cohomology is defined by

$$(3.5) \quad H_{(2)}^n(S, \mathbf{V}) = \frac{\{\eta \in \mathcal{A}^n(S, \mathbf{V}): \eta \text{ is } L_2 \text{ and } d\eta = 0\}}{\{\eta \text{ as above: } \eta = d\psi \text{ for some } L_2 \psi \in \mathcal{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

$$(3.6) \quad H_{(2)}^n(S, \mathbf{V}) \rightarrow H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) *Remark.* We may compute the L_2 cohomology groups (3.5) from the complex of weakly differentiable L_2 forms $\mathcal{L}_{(2)}^*(S, \mathbf{V})$; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then d becomes a densely-defined differential for the “complex” of Hilbert spaces of \mathbf{V} -valued L_2 forms, and

$$H_{(2)}^n(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed } \mathbf{V}\text{-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_2(n-1)\text{-forms}\}}.$$

We define the *reduced* L_2 cohomology $\bar{H}_{(2)}^n(S, \mathbf{V})$ by replacing the range of d in the above quotient by its Hilbert space closure; the reduced L_2 cohomology inherits a Hilbert space structure from the L_2 inner product.

In discussing $\|\eta\|_{(2)}$, we wish to make use of the form $\tilde{\eta}$ of (3.4), and we have

(3.8) **LEMMA** [7, p. 380]. If $\eta \in \mathcal{A}^n(S, \mathbf{V})$ and $\tilde{\eta} \in \mathcal{A}^n(\Gamma \backslash G) \otimes V$ is the corresponding element, then

$$\|\eta\|_{(2)}^2 = c \|\tilde{\eta}\|_{(2)}^2,$$

where c equals the volume of K .

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis $\{X_i\}_{i=1}^k$ of \mathfrak{p}^+ , so

$$\{X_1, \bar{X}_1, \dots, X_k, \bar{X}_k\}$$

forms an orthonormal basis of $\mathfrak{p}_{\mathbb{C}}$. For $\eta \in \mathcal{A}^{p,q}(S, V)$, put

$$\eta_{i_1, \dots, i_p; j_1, \dots, j_q} = \tilde{\eta}(X_{i_1}, \dots, X_{i_p}, \bar{X}_{j_1}, \dots, \bar{X}_{j_q}) \in \mathcal{A}^0(G) \otimes V.$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative d on $\mathcal{A}^\bullet(S, V)$ into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators D' and d'_p are defined by the formulas

$$\begin{aligned} (3.9) \quad & (D'\eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \dots, \widehat{i_u}, \dots, i_{p+1}; j_1, \dots, j_q}, \end{aligned}$$

$$\begin{aligned} (3.10) \quad & (d'_p \eta)_{i_1, \dots, i_{p+1}; j_1, \dots, j_q} \\ &= \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_u}) \eta_{i_1, \dots, \widehat{i_u}, \dots, i_{p+1}; j_1, \dots, j_q}. \end{aligned}$$

One also puts $D'' = \overline{D'}$ and $d''_p = \overline{d'_p}$. Then $d' = D' + d'_p$ and $d'' = D'' + d''_p$; if we put $D = D' + D''$ and $d_p = d'_p + d''_p$, then $d = D + d_p$. We remark that D gives a metric connection on $\Phi(\rho)$; heuristically, we regard $\kappa^*E(\rho)$ as being canonically flat.

Let \mathfrak{D} represent any of the above operators. One can obtain directly formulas for the L_2 adjoint \mathfrak{D}^* and the Laplacian

$$(3.11) \quad \square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) PROPOSITION. As operators on $\mathcal{A}^\bullet(S, V)$,

- i) $\square_d = \square_{d'} + \square_{d''}$
- ii) $\square_d = \square_D + \square_{d_p}$
- iii) $\square_D = \square_{D'} + \square_{D''}$
- iv) $\square_{d_p} = \square_{d'_p} + \square_{d''_p}$
- v) $\square_{d'} = \square_{D'} + \square_{d'_p}$

(3.13) Remark. One always has

$$\square_{(\mathfrak{D}_1 + \mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1 \mathfrak{D}_2^* + \mathfrak{D}_2^* \mathfrak{D}_1 + \mathfrak{D}_1^* \mathfrak{D}_2 + \mathfrak{D}_2 \mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since S is complete in the induced metric from M , the operators \mathfrak{D} as above have unique [3] closed extensions to $\mathcal{L}_{(2)}^\bullet(S, V)$, so the identities (3.12) continue to remain valid in the strict sense on L_2 . From this, one may conclude

(3.14) PROPOSITION. If $\eta \in \mathcal{L}_{(2)}^\bullet(S, V)$, the following are equivalent:

- i) $\square_d \eta = 0$ (η is harmonic),
- ii) $\square_{d'} \eta = \square_{d''} \eta = 0$
- iii) $\square_{D'} \eta = \square_{D''} \eta = \square_{d'_p} \eta = \square_{d''_p} \eta = 0$,
- iv) $D' \eta = (D')^* \eta = D'' \eta = (D'')^* \eta = d'_p \eta$
 $= (d'_p)^* \eta = d''_p \eta = (d''_p)^* \eta = 0$.

Since $\square_{\mathfrak{D}}$ is elliptic for any of the operators \mathfrak{D} above, harmonic forms are necessarily C^∞ . Let $\mathcal{H}_{(2)}^n(S, V)$ denote the space of L_2 harmonic n -forms with values in V . We obtain by standard theory (see [15, §1]):

(3.15) PROPOSITION. For all n ,

- i) $\bar{H}_{(2)}^n(S, V) \simeq \mathcal{H}_{(2)}^n(S, V)$,
- ii) The mapping $\mathcal{H}_{(2)}^n(S, V) \rightarrow H_{(2)}^n(S, V)$ is injective, and is an isomorphism if and only if d , operating on $\mathcal{L}_{(2)}^{n-1}(S, V)$, has closed range.

(3.16) *Remark.* An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that $H_{(2)}^n(S, V)$ is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators d' and d'' . Therefore, a form is harmonic if and only if its (p, q) components are harmonic, so

$$(3.17) \quad \mathcal{H}_{(2)}^n(S, V) = \bigoplus_{p+q=n} \mathcal{H}_{(2)}^{p,q}(S, V).$$

Passing this through the isomorphism (3.15, i), we get

$$(3.18) \quad \bar{H}_{(2)}^n(S, V) = \bigoplus_{p+q=n} H_{(2)}^{p,q}(S, V).$$

If we take S to be compact, we have $H_{(2)}^n(S, V) = H^n(S, V)$, and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) PROPOSITION [8, p. 14].

$$\square_{D''} + \square_{d'_p} = \square_{D'} + \square_{d''_p}.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY. η is harmonic if and only if

$$\square_{D''}\eta = \square_{d'_p}\eta = 0.$$

We close this section with a brief account of another way of viewing the cohomology groups $H^n(\Gamma; \rho, V)$, currently preferred in representation theory. For simplicity, we assume that S is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of $\mathcal{A}^n(S, V)$ as a mapping from $\Lambda^n \mathfrak{p}_C$ into $\mathcal{A}^0(\Gamma \backslash G) \otimes V$ that satisfies a transformation rule under \mathfrak{k} . This correspondence gives an isomorphism of $H^n(S, V)$ with the *relative Lie algebra cohomology* (see, e.g. [8, pp. 6-8] or [14, Ch. I]):

$$(3.21) \quad H^n(\mathfrak{g}_C, \mathfrak{k}_C, \mathcal{A}^0(\Gamma \backslash G) \otimes V),$$

associated to the cochain complex

$$(3.22) \quad \text{Hom}_K (\Lambda^* \mathfrak{p} \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

Here, $\mathfrak{g}_\mathbb{C}$ acts on $\mathcal{A}^0 (\Gamma \backslash G)$ by differentiation, induced by the regular representation of G .

(3.23) *Remark.* By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H_d^n (G, \mathcal{A}^0 (\Gamma \backslash G) \otimes V).$$

For this reason, (3.21) is often referred to as "continuous cohomology."

The cohomology (3.21) decomposes according to the splitting of $\mathcal{A}^0 (\Gamma \backslash G) \otimes V$. First, one decomposes $L_2 (\Gamma \backslash G)$ as a representation of G :

$$(3.24) \quad L_2 (\Gamma \backslash G) \simeq \widehat{\bigoplus}_{\alpha} E_{\alpha}$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

$$(3.25) \quad L_2 (\Gamma \backslash G, V) \simeq \widehat{\bigoplus}_{\alpha} (E_{\alpha} \otimes V)$$

Taking C^∞ vectors gives the decomposition

$$(3.26) \quad \mathcal{A}^0 (\Gamma \backslash G) \otimes V \simeq \widehat{\bigoplus}_{\alpha} (E_{\alpha}^{\infty} \otimes V),$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form $\tilde{\eta}$, the Laplacian is given by

$$(3.27) \quad \widetilde{\square \eta} = [-C + \rho(C)] \tilde{\eta},$$

where C is the Casimir element of the enveloping algebra of \mathfrak{g} . It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters χ_{α} of $(\pi_{\alpha}, E_{\alpha})$ and χ_{ρ} of (ρ, V) agree on C . In fact, if the space of harmonic forms is non-zero one must have $\chi_{\alpha} = \chi_{\rho}$ (see [1, (2.4)]). In this case, every cochain with values in E_{α} is harmonic. Thus,

$$(3.28) \quad \begin{aligned} H^n (S, V) &\simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} \text{Hom}_K (\Lambda^n \mathfrak{p}_{\mathbb{C}}, E_{\alpha} \otimes V) \\ &\simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} (\Lambda^n \mathfrak{p}_{\mathbb{C}}^* \otimes E_{\alpha} \otimes V)^K \quad (K\text{-invariants}). \end{aligned}$$

From (3.27) and (3.28), one obtains the following:

(3.29) PROPOSITION. Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible representations of G , and suppose that $\rho_1(C) = \rho_2(C)$. Then every morphism of K -representations

$$\phi: \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \rightarrow \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_*: \mathcal{H}^{n_1}(S, V_1) \rightarrow \mathcal{H}^{n_2}(S, V_2).$$

and thus a mapping $\phi_*: H^{n_1}(S, V_1) \rightarrow H^{n_2}(S, V_2)$. (If the infinitesimal characters of (ρ_1, V_1) and (ρ_2, V_2) differ, then ϕ_* is the zero mapping.)

If we now decompose each $\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V$ as a representation of K and apply (3.29) to the projections onto each component, there is induced decomposition of $H^n(S, V)$, much in the spirit of [2]. If we decompose only $\Lambda^n \mathfrak{p}^*$, we obtain the decomposition (3.18). We will refine that decomposition in §5.

If S is non-compact, then $L_2(\Gamma \backslash G)$ is the direct sum of its discrete spectrum $L_2(\Gamma \backslash G)_d$ and the continuous spectrum $L_2(\Gamma \backslash G)_{ct}$. One then has a decomposition like (3.24) only for $L_2(\Gamma \backslash G)_d$. From there, one obtains an injection

$$(3.30) \quad \bigoplus_{\alpha} (E_{\alpha}^{\infty} \otimes V) \rightarrow \mathcal{A}_{(2)}^0(\Gamma \backslash G) \otimes V,$$

whose image consists of those C^∞ V -valued functions for which all left-invariant differential operators are in L_2 . Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if Γ is an arithmetic subgroup of G , then all harmonic forms come from $L_2(\Gamma \backslash G)_d$. In this case, one therefore obtains, as in (3.28), the isomorphism

$$(3.31) \quad \bar{H}_{(2)}^n(S, V) \simeq \bigoplus_{\chi_\alpha = \chi_p} (\Lambda^n \mathfrak{p}_\mathbb{C}^* \otimes E_\alpha \otimes V)^K.$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced L_2 cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced) L_2 cohomology is for some groups infinite-dimensional, with d having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein ¹⁾. (See also [12] for a different approach to the L_2 cohomology.)

¹⁾ See note added in proof.