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§3. The cohomology groups $H^n(\Gamma; \rho, V)$

In this section, we will discuss the various approaches toward computing the Eilenberg-MacLane cohomology groups $H^n(\Gamma; \rho, V)$ for a finite-dimensional representation (ρ, V) of G, which we may as well take to be irreducible.

We begin with the use of deRham cohomology, as carried out originally in [7]. Since M is contractible, there is a natural isomorphism

$$H^n(\Gamma; \rho, V) \simeq H^n(S, \mathbf{V})$$

(with notation as in §2), hence we may compute these cohomology groups from the complex of V-valued C^{∞} forms on S (by the deRham theorem).

We will make use of the following obvious diagram of manifolds

(3.1)
$$G \xrightarrow{\psi} \Gamma \backslash G$$

$$\kappa \downarrow \qquad \downarrow \qquad \lambda$$

$$M \xrightarrow{\pi} S$$

Let η be an element of $\mathscr{A}^n(S, \mathbf{V})$, the space of global C^{∞} n-forms on M with values in \mathbf{V} . Then

$$\phi = \kappa^* \pi^* \eta$$

is a V-valued form on G satisfying the equations

(3.2) i)
$$\gamma^* \phi = \rho(\gamma) \phi$$
 if $\gamma \in \Gamma$
ii) $\mathcal{L}_{\gamma} \phi = 0$ if $Y \in \mathfrak{k}$,

$$\mathcal{L}_{\gamma} = \text{Lie derivative} = (\Lambda^n \text{Ad*})(Y)$$
iii) $\iota_{\gamma} \phi = 0$ if $Y \in \mathfrak{k}$

$$\iota_{\gamma} = \text{interior multiplication by } Y$$

Conversely, every element $\phi \in \mathscr{A}^n(G) \otimes_{\mathbf{C}} V(\mathscr{A}^n(G))$ denoting the space of C^{∞} n-forms on G) that satisfies (3.2) is $\kappa^*\pi^*\eta$ for some $\eta \in \mathscr{A}^n(S, \mathbf{V})$. We then apply the mapping Ξ of (2.6) to ϕ , obtaining the *n*-form

$$\tilde{\eta} = \rho (g^{-1}) \phi$$

which satisfies

(3.4) i)
$$\gamma^* \tilde{\eta} = \tilde{\eta}$$
 if $\gamma \in \Gamma$,
ii) $\mathcal{L}_{\gamma} \tilde{\eta} = -\rho(Y) \tilde{\eta}$ if $Y \in \tilde{t}$,
iii) $\iota_{\gamma} \tilde{\eta} = 0$ if $Y \in \tilde{t}$.

In particular, we may view $\tilde{\eta}$ as a vector-valued form on $\Gamma \backslash G$.

We next describe the Hodge theory for $H^n(S, \mathbf{V})$ from this point of view, as was done in [7] and [8]. Actually, one must work with the L_2 cohomology when S is non-compact. Since we have defined a metric on $A(\Gamma, \rho)$ in Section 2, and on the tangent bundle by the Killing form, there is an L_2 norm $\|\eta\|_{(2)}$ for $\eta \in \mathscr{A}^n(S, \mathbf{V})$, and the L_2 cohomology is defined by

$$(3.5)$$

$$H_{(2)}^{n}(S, \mathbf{V}) = \frac{\{\eta \in \mathscr{A}^{n}(S, \mathbf{V}): \eta \text{ is } L_{2} \text{ and } d\eta = 0\}}{\{\eta \text{ as above: } \eta = d\psi \text{ for some } L_{2} \text{ } \psi \in \mathscr{A}^{n-1}(S, \mathbf{V})\}}$$

There is then an obvious mapping

$$(3.6) H_{(2)}^n(S, \mathbf{V}) \to H^n(S, \mathbf{V}),$$

and one is ultimately interested in understanding the kernel and image of this mapping. (See also [12].)

(3.7) Remark. We may compute the L_2 cohomology groups (3.5) from the complex of weakly differentiable L_2 forms $\mathcal{L}^{\bullet}_{(2)}(S, \mathbf{V})$; i.e., we may drop the smoothness condition on forms (see [15, §8]). Then d becomes a densely-defined differential for the "complex" of Hilbert spaces of \mathbf{V} -valued L_2 forms, and

$$H_{(2)}^{n}(S, \mathbf{V}) \simeq \frac{\{\text{weakly closed V-valued } n\text{-forms}\}}{\{\text{range of } d \text{ on } L_{2}(n-1)\text{-forms}\}}.$$

We define the reduced L_2 cohomology $\overline{H}_{(2)}^n(S, V)$ by replacing the range of d in the above quotient by its Hilbert space closure; the reduced L_2 cohomology inherits a Hilbert space structure from the L_2 inner product.

In discussing $\|\eta\|_{(2)}$, we wish to make use of the form $\tilde{\eta}$ of (3.4), and we have

(3.8) LEMMA [7, p. 380]. If $\eta \in \mathscr{A}^n(S, V)$ and $\tilde{\eta} \in \mathscr{A}^n(\Gamma \setminus G) \otimes V$ is the corresponding element, then

$$\| \eta \|_{(2)}^2 = c \| \tilde{\eta} \|_{(2)}^2$$

where c equals the volume of K.

While much of what follows holds in the absence of a complex structure, we restrict ourselves to the Hermitian symmetric case for the purposes of this exposition. For the general case see [7].

Choose an orthonormal basis $\{X_i\}_{i=1}^k$ of \mathfrak{p}^+ , so

$$\{X_1, \bar{X}_1, ..., X_k, \bar{X}_k\}$$

forms an orthonormal basis of $\mathfrak{p}_{\mathbf{c}}$. For $\eta \in \mathscr{A}^{p, q}(S, \mathbf{V})$, put

$$\eta_{i_1, \, \dots, \, i_p; \, j_1, \, \dots, \, j_q} = \tilde{\eta} (X_{i_1, \, \dots, \,} X_{i_p}, \bar{X}_{j_1, \, \dots, \,} \bar{X}_{j_q}) \in \mathcal{A}^0 (G) \otimes V$$

Let

$$d = d' + d''$$

be the usual decomposition of the (flat) exterior derivative d on $\mathscr{A}^{\bullet}(S, \mathbf{V})$ into components of bidegree (1, 0) and (0, 1). The bidegree (1, 0) differential operators D' and d'_p are defined by the formulas

(3.9)
$$(D'\eta)_{i_1, \ldots, i_{p+1}; i_1, \ldots, j_q}$$

$$= \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta_{i_1, \ldots, \widehat{i_u}, \ldots, i_{p+1}; j_1, \ldots, j_q},$$

(3.10)
$$(d'_{\rho}\eta)_{i_{1}, \dots, i_{p+1}; j_{1}, \dots, j_{q}}$$

$$= \sum_{u=1}^{p+1} (-1)^{u-1} \rho(X_{i_{u}}) \eta_{i_{1}, \dots, \widehat{i_{u}}, \dots, i_{p+1}; j_{1}, \dots, j_{q}}.$$

One also puts $D'' = \overline{D'}$ and $d''_{\rho} = \overline{d'_{\rho}}$. Then $d' = D' + d'_{\rho}$ and $d'' = D'' + d''_{\rho}$; if we put D = D' + D'' and $d_{\rho} = d'_{\rho} + d''_{\rho}$, then $d = D + d_{\rho}$. We remark that D gives a metric connection on $\Phi(\rho)$; heuristically, we regard $\kappa^*E(\rho)$ as being canonically flat.

Let \mathfrak{D} represent any of the above operators. One can obtain directly formulas for the L_2 adjoint \mathfrak{D}^* and the Laplacian

$$\square_{\mathfrak{D}} = \mathfrak{D}\mathfrak{D}^* + \mathfrak{D}^*\mathfrak{D}$$

(see [9, pp. 68-70]). From these calculations, one obtains also the following identities

(3.12) Proposition. As operators on $\mathscr{A}^{\bullet}(S, V)$,

i)
$$\square_d = \square_{d'} + \square_{d''}$$

ii)
$$\square_d = \square_D + \square_{d_0}$$

iii)
$$\square_D = \square_{D'} + \square_{D''}$$

iv)
$$\square_{d_o} = \square_{d'_o} + \square_{d''_o}$$

$$\mathbf{v}) \square_{d'} = \square_{D'} + \square_{d'_{\mathbf{0}}}$$

(3.13) Remark. One always has

$$\square_{(\mathfrak{D}_1 + \mathfrak{D}_2)} = \square_{\mathfrak{D}_1} + \square_{\mathfrak{D}_2} + (\mathfrak{D}_1 \mathfrak{D}_2^* + \mathfrak{D}_2^* \mathfrak{D}_1 + \mathfrak{D}_1^* \mathfrak{D}_2 + \mathfrak{D}_2 \mathfrak{D}_1^*),$$

so (3.12) amounts to establishing the vanishing of the expression in parentheses on the right-hand side. The identities in (3.12) are *not* general formulas for flat bundles on manifolds, but are particular to the group-theoretic context.

Since S is complete in the induced metric from M, the operators \mathfrak{D} as above have unique [3] closed extensions to $\mathscr{L}^{\bullet}_{(2)}(S, \mathbf{V})$, so the identities (3.12) continue to remain valid in the strict sense on L_2 . From this, one may conclude

(3.14) Proposition. If $\eta \in \mathcal{L}_{(2)}^{\bullet}(S, \mathbf{V})$, the following are equivalent:

i)
$$\Box_d \eta = 0$$
 (η is harmonic),

ii)
$$\square_{d'} \eta = \square_{d''} \eta = 0$$

iii)
$$\square_{D'} \eta = \square_{D''} \eta = \square_{d'_{\rho}} \eta = \square_{d'_{\rho}} \eta = 0,$$

iv)
$$D'\eta = (D')^*\eta = D''\eta = (D'')^*\eta = d'_{\rho}\eta$$

= $(d'_{\rho})^*\eta = d''_{\rho}\eta = (d''_{\rho})^*\eta = 0$.

Since $\square_{\mathfrak{D}}$ is elliptic for any of the operators \mathfrak{D} above, harmonic forms are necessarily C^{∞} . Let $\mathscr{L}^n_{(2)}(S, \mathbf{V})$ denote the space of L_2 harmonic *n*-forms with values in \mathbf{V} . We obtain by standard theory (see [15, §1]):

(3.15) Proposition. For all n,

i)
$$\bar{H}_{(2)}^{n}(S, \mathbf{V}) \simeq \mathcal{L}_{(2)}^{n}(S, \mathbf{V}),$$

ii) The mapping $\mathcal{L}_{(2)}^n(S, \mathbf{V}) \to H_{(2)}^n(S, \mathbf{V})$ is injective, and is an isomorphism if and only if d, operating on $\mathcal{L}_{(2)}^{n-1}(S, \mathbf{V})$, has closed range.

(3.16) Remark. An easy way to guarantee that the mapping in (3.15, ii) is an isomorphism is by showing that $H_{(2)}^n(S, \mathbf{V})$ is finite-dimensional.

By (3.14, ii) a form is harmonic if and only if it is annihilated by the Laplacians of the bidegree-preserving operators d' and d''. Therefore, a form is harmonic if and only if its (p, q) components are harmonic, so

(3.17)
$$\mathscr{K}_{(2)}^{n}(S, \mathbf{V}) = \bigoplus_{p+q=n} \mathscr{K}_{(2)}^{p,q}(S, \mathbf{V}).$$

Passing this through the isomorphism (3.15, i), we get

(3.18)
$$\bar{H}_{(2)}^{n}(S, \mathbf{V}) = \bigoplus_{p+q=n} H_{(2)}^{p, q}(S, \mathbf{V}).$$

If we take S to be compact, we have $H_{(2)}^n(S, \mathbf{V}) = H^n(S, \mathbf{V})$, and in (3.18) the Hodge decomposition of [7].

The most significant assertion about Laplacians, as we will see in Section 5, is given by

(3.19) Proposition [8, p. 14].

$$\square_{D^{\prime\prime}} + \square_{d_{\rho}^{\prime}} = \square_{D^{\prime}} + \square_{d_{\rho}^{\prime\prime}}.$$

This fact was not fully exploited in the earlier work.

(3.20) COROLLARY. η is harmonic if and only if

$$\square_{\textit{D}^{\prime\prime}} \eta \; = \; \square_{\textit{d}_{\textit{p}}^{\prime}} \eta \; = \; 0 \; .$$

We close this section with a brief account of another way of viewing the cohomology groups $H^n(\Gamma; \rho, V)$, currently preferred in representation theory. For simplicity, we assume that S is compact, and mention at the end what changes must be made in the non-compact case.

From the description (3.4), it is clear that we may regard an element of $\mathcal{A}^n(S, \mathbf{V})$ as a mapping from $\Lambda^n \mathbf{p_C}$ into $\mathcal{A}^0(\Gamma \backslash G) \otimes V$ that satisfies a transformation rule under f. This correspondence gives an isomorphism of $H^n(S, \mathbf{V})$ with the relative Lie algebra cohomology (see, e.g. [8, pp. 6-8] or [14, Ch. Π):

$$(3.21) Hn (gC, fC, \mathscr{A}0 (\Gamma \backslash G) \otimes V),$$

associated to the cochain complex

(3.22)
$$\operatorname{Hom}_{K}\left(\Lambda^{\bullet}\mathfrak{p}\,\mathscr{A}^{0}\left(\Gamma\backslash G\right)\otimes V\right).$$

Here, $g_{\mathbb{C}}$ acts on $\mathscr{A}^0(\Gamma \backslash G)$ by differentiation, induced by the regular representation of G.

(3.23) Remark. By a theorem of van Est (see [5, p. 386]), the relative Lie algebra cohomology is in turn isomorphic to the differentiable (or even continuous) Eilenberg-MacLane cohomology

$$H_d^n(G, \mathcal{A}^0(\Gamma \backslash G) \otimes V)$$
.

For this reason, (3.21) is often referred to as "continuous cohomology."

The cohomology (3.21) decomposes according to the splitting of \mathscr{A}^0 ($\Gamma \backslash G$) $\otimes V$. First, one decomposes L_2 ($\Gamma \backslash G$) as a representation of G:

$$(3.24) L_2(\Gamma \backslash G) \simeq \bigoplus_{\alpha} E_{\alpha}$$

into the direct sum of irreducible unitary representations of finite multiplicity. Then

(3.25)
$$L_2(\Gamma \backslash G, V) \simeq \bigoplus_{\alpha} (E_{\alpha} \otimes V)$$

Taking C^{∞} vectors gives the decomposition

(3.26)
$$\mathscr{A}^{0}\left(\Gamma\backslash G\right)\otimes V\simeq \bigoplus_{\alpha}\left(E_{\alpha}^{\infty}\otimes V\right),$$

By a formula of Kuga (see [7, p. 385] or [14, p. 49]), in terms of the form $\tilde{\eta}$, the Laplacian is given by

$$(3.27) \qquad \qquad \boxed{ \Box \eta} = [-C + \rho(C)] \tilde{\eta} ,$$

where C is the Casimir element of the enveloping algebra of g. It follows that in each summand of (3.26), there can be non-zero harmonic forms only if the infinitesimal characters χ_{α} of $(\pi_{\alpha}, E_{\alpha})$ and χ_{ρ} of (ρ, V) agree on C. In fact, if the space of harmonic forms is non-zero one must have $\chi_{\alpha} = \chi_{\rho}$ (see [1, (2.4)]). In this case, every cochain with values in E_{α} is harmonic. Thus,

(3.28)
$$H^{n}(S, \mathbf{V}) \simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} \operatorname{Hom}_{K}(\Lambda^{n}\mathfrak{p}_{\mathbf{C}}, E_{\alpha} \otimes V)$$
$$\simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} (\Lambda^{n}\mathfrak{p}_{\mathbf{C}}^{*} \otimes E_{\alpha} \otimes V)^{K} \quad (K\text{-invariants}).$$

From (3.27) and (3.28), one obtains the following:

(3.29) Proposition. Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible representations of G, and suppose that $\rho_1(C) = \rho_2(C)$. Then every morphism of K-representations

$$\phi: \Lambda^{n_1} \mathfrak{p}^* \otimes V_1 \to \Lambda^{n_2} \mathfrak{p}^* \otimes V_2$$

induces a mapping of harmonic forms

$$\phi_* : \mathcal{L}^{n_1}(S, \mathbf{V}_1) \to \mathcal{L}^{n_2}(S, \mathbf{V}_2).$$

and thus a mapping $\phi_*: H^{n_1}(S, \mathbf{V}_1) \to H^{n_2}(S, \mathbf{V}_2)$. (If the infinitesimal characters of (ρ_1, V_1) and (ρ_2, V_2) differ, then ϕ_* is the zero mapping.)

If we now decompose each $\Lambda^n \mathfrak{p}_{\mathbb{C}}^* \otimes E_{\alpha} \otimes V$ as a representation of K and apply (3.29) to the projections onto each component, there is induced decomposition of $H^n(S, V)$, much in the spirit of [2]. If we decompose only $\Lambda^n \mathfrak{p}^*$, we obtain the decomposition (3.18). We will refine that decomposition in §5.

If S is non-compact, then $L_2(\Gamma \backslash G)$ is the direct sum of its discrete spectrum $L_2(\Gamma \backslash G)_d$ and the continuous spectrum $L_2(\Gamma \backslash G)_{ct}$. One then has a decomposition like (3.24) only for $L_2(\Gamma \backslash G)_d$. From there, one obtains an injection

$$(3.30) \qquad \qquad \bigoplus_{\alpha} (E_{\alpha}^{\infty} \otimes V) \to \mathscr{A}_{(2)}^{0} (\Gamma \backslash G) \otimes V ,$$

whose image consists of those C^{∞} V-valued functions for which all left-invariant differential operators are in L_2 . Borel has shown that (3.30) induces an isomorphism on cohomology. Also, if Γ is an arithmetic subgroup of G, then all harmonic forms come from L_2 ($\Gamma \setminus G$)_d. In this case, one therefore obtains, as in (3.28), the isomorphism

(3.31)
$$\bar{H}_{(2)}^{n}(S, \mathbf{V}) \simeq \bigoplus_{\chi_{\alpha} = \chi_{\rho}} (\Lambda^{n} \mathfrak{p}_{\mathbf{C}}^{*} \otimes E_{\alpha} \otimes V)^{K}.$$

Moreover, the above sum has only finitely many non-zero terms, as the reduced L_2 cohomology is finite-dimensional. Borel discovered the initially surprising phenomenon that the (non-reduced) L_2 cohomology is for some groups infinite-dimensional, with d having non-closed range on the continuous spectrum in certain dimensions; however, this never occurs in the Hermitian case. As a reference for this paragraph, see [13] and the references cited therein 1). (See also [12] for a different approach to the L_2 cohomology.)

¹⁾ See note added in proof.