Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE

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Kapitel: §2. Vector bundles on \M

DOI: https://doi.org/10.5169/seals-51751

§2. Vector bundles on $\Gamma\backslash M$

Let Γ be a discrete subgroup of G which acts freely on the symmetric space M, and put $S = \Gamma\backslash M$. We will discuss two standard constructions of vector bundles on S.

The first type is the quotient by Γ of a homogeneous vector bundle on M. Specifically, let (τ, W) be a finite-dimensional representation of K. Then $E(\tau)$ is defined as the quotient of $G \times W$ by the following identification under the action of K:

$$(2.1) \qquad (g, w) \sim \left(gk^{-1}, \tau(k)\, w\right) \quad \text{if} \quad k \in K .$$

$E(\tau)$ is naturally a C^∞ vector bundle on M, and the left action of G on M is covered by the obvious left action of G on $E(\tau)$. Thus, we may take the quotient by any Γ as above to obtain a bundle $E(\Gamma, \tau)$ on S. Alternatively, $E(\Gamma, \tau)$ is an associated vector bundle of the principal K-bundle $\Gamma\backslash G$. Note that if (τ, W) decomposes as a representation of K into

$$(\tau, W) = \oplus_{i=1}^{l} (\tau_i, W_i) ,$$

then one gets an induced decomposition

$$(2.2) \qquad E(\Gamma, \tau) \simeq \oplus_{i=1}^{l} E(\Gamma, \tau_i) .$$

We may identify sections of $E(\Gamma, \tau)$ as the Γ-invariant sections of $E(\tau)$, which in turn are given by mappings $\phi : G \to W$ which satisfy

$$(2.3) \qquad \phi(\gamma g k^{-1}) = \tau(k)\, \phi(g) \quad \text{for all} \quad \gamma \in \Gamma,\ g \in G,\ k \in K .$$

An Hermitian metric can be placed on $E(\Gamma, \tau)$ by a choice of $\tau(K)$-invariant inner product on W. (Such exist because K is compact.) The corresponding constant metric on $G \times W$ descends to $E(\Gamma, \tau)$, in view of (2.1).

(2.4) *Example.* Taking $\tau = \mathrm{Ad}\,|_{\mathfrak{p}_{\mathbf{C}}}$, we have a natural isomorphism of $E(\tau)$ and the complexified tangent bundle to M, and we may take quotients by Γ.

The second type of vector bundle is the flat bundle associated to a finite-dimensional representation (ψ, V) of Γ. We let $\Phi(\psi)$ denote the quotient of $M \times V$ under the action of Γ:

$$(m, v) \sim \left(\gamma m, \psi(\gamma)\, v\right) .$$

Sections of $\Phi(\psi)$ are given by functions $f : M \to V$ such that

$$f(\gamma x) = \psi(\gamma) f(x) \quad \text{if} \quad \gamma \in \Gamma,\ x \in M. \tag{2.5}$$

The local sections of $\Phi(\psi)$ determined by constant V-valued functions determine a flat structure on $\Phi(\psi)$, whose sheaf of locally constant sections will be denoted $\mathbf{V}$.

The two constructions above are related by the elementary

(2.6) PROPOSITION. *Let* (ρ, V) *be a representation of* G *(which then restricts to representations of* K *and* Γ*). Then the mapping*

$$\tilde{\Xi}: G \times V \to G \times V,$$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, *induces an isomorphism of* C^∞ *vector bundles*

$$\Xi: \Phi(\rho|_\Gamma) \xrightarrow{\sim} E(\Gamma, \rho|_K).$$

(2.7) *Remark.* Let (ρ, V) be a finite-dimensional representation of G, and (ψ, W) a finite-dimensional *unitary* representation of Γ. We note that by the standard ruse of replacing G by $G' = G \times U(W)$, where $U(W)$ denotes the unitary group of W, $V \otimes W$ becomes a representation space for G', and so the bundle $\Phi(\rho|_\Gamma \otimes \psi)$ falls into the class of bundles covered by (2.6).

A natural metric on $\Phi(\rho|_\Gamma)$ is provided by the admissible inner product T (1.9). For $g \in G$, $v, w \in V$, let (at $gx_0 \in M$)

$$\langle v, w \rangle_{gx_0} = T\left(\rho(g^{-1}) v, \rho(g^{-1}) w\right). \tag{2.8}$$

Since K acts isometrically with respect to T, it follows that (2.8) is well-defined on $M \times V$; and it is evident that the action of Γ is isometric, so (2.8) descends to $\Phi(\rho|_\Gamma)$. T also determines a metric in $E(\Gamma, \rho|_K)$, and it is clear that the mapping Ξ of (2.6) is then an isometry of bundles.

Assume next that M is Hermitian. Then to every finite-dimensional holomorphic representation (σ, W) of Q is associated a $G_{\mathbf{C}}$-equivariant holomorphic vector bundle $\check{E}(\sigma)$ on $\check{M}$, constructed as in (2.1). By restricting to M and taking the quotient by the action of Γ, we obtain the holomorphic vector bundle $\check{E}(\Gamma, \sigma)$ on S. Q-invariant subspaces of W determine holomorphic sub-bundles of $\check{E}(\Gamma, \sigma)$. Along the same lines as (2.6), we have:

(2.9) PROPOSITION. Let (ρ, V) be a representation of G (which then determines representations of Q and Γ). Then the mapping

$$\tilde{\Xi}: G_{\mathbf{C}} \times V \to G_{\mathbf{C}} \times V,$$

defined by $\tilde{\tilde{\Xi}}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of holomorphic bundles

$$\Xi : \Phi(\rho|_\Gamma) \xrightarrow{\sim} \check{E}(\Gamma, \rho_{\mathbf{C}}|_Q).$$

Every representation τ of K determines a holomorphic representation of $K_{\mathbf{C}}$, which then extends to a representation σ_τ of Q by setting σ_τ to be trivial on P^-, since K normalizes P^-. The C^∞ isomorphism $E(\Gamma, \tau) \to \check{E}(\Gamma, \sigma_\tau)$ imparts a holomorphic structure to $E(\Gamma, \tau)$; however, an isomorphism (2.2) need not be holomorphically compatible with (2.9).

(2.10) *Example.* Taking $\tau = \mathrm{Ad}^+ = \mathrm{Ad}\, K|_{\mathfrak{p}^+}$ we obtain a holomorphic isomorphism

$$E(\tau) \simeq \Theta_M \qquad \text{(holomorphic tangent bundle of } M),$$

and we may take quotients by Γ. Therefore, since the Killing form gives $(\mathfrak{p}^+)^* \simeq \mathfrak{p}^-$ as a representation of K,

$$E(\Gamma, \Lambda^p \mathrm{Ad}^-) \simeq \Omega^p_S.$$

(Here and elsewhere, we identify a vector bundle with its locally free sheaf of germs of sections.)

There is a relation of the preceding to automorphic forms, coming from the following. Let W be a finite dimensional vector space over $\mathbf{C}$. Then an *automorphy factor* $\mathscr{J}$ is a C^∞ mapping

$$\mathscr{J} : G \times M \to GL(W)$$

which satisfies

(2.11) i) $\mathscr{J}(g, x)$ is, for fixed g, a holomorphic mapping from M into $GL(W)$,
ii) $\mathscr{J}(gh, x) = \mathscr{J}(g, hx)\mathscr{J}(h, x)$.

We observe that $\mathscr{J}$ is then completely determined by the function $\mathscr{J}(g, x_0)$ on G. Given such a $\mathscr{J}$, one forms the *automorphic vector bundle* $A(\Gamma, \mathscr{J})$, a holomorphic bundle, by taking the quotient of $M \times W$ under the action of Γ:

$$(x, w) \sim (\gamma x, \mathscr{J}(\gamma, x) w) \qquad \text{for all} \quad \gamma \in \Gamma,\ x \in M,\ w \in W.$$

Sections of $A(\Gamma, \mathscr{J})$ are then given by functions $f : M \to W$ such that

$$f(\gamma x) = \mathscr{J}(\gamma, x) f(x) \qquad \text{for all} \quad \gamma \in \Gamma,\ x \in M; \tag{2.12}$$

these are called *automorphic forms.*

From an automorphy factor j, one obtains a representation τ_j of K by setting

$$\tau_j(k) = j(k, x_0),$$

because of (2.11, ii). We then have

(2.13) PROPOSITION. *Let j be an automorphy factor. Then there is a C^∞ isomorphism*

$$\Psi : E(\Gamma, \tau_j) \to A(\Gamma j),$$

induced by the mapping

$$\tilde{\Psi} : G \times W \to G \times W$$
$$\tilde{\Psi}(g, w) = \bigl(g, j(g, x_0)\, w\bigr).$$

(2.14) *Remark.* For a representation (ρ, V) of G,

$$j(g, x) = \rho(g)$$

defines an automorphy factor, for which (2.13) is a reformulation of (2.6).

Conversely, to the Lie group G is associated its *canonical automorphy factor* $\mathcal{J}$ (see [7, p. 397]), which is a C^∞ mapping $\mathcal{J} : G \times M \to K_{\mathbf{C}}$ which satisfies the equations of (2.11); and $\mathcal{J}(g, x_0)$ is the $K_{\mathbf{C}}$-component of g in $G \subset U = P^+ K_{\mathbf{C}} P^-$. Then each representation τ of K determines an automorphy factor

$$j_\tau(g, x) = \tau\bigl(\mathcal{J}(g, x)\bigr).$$

In this case, the mapping $\tilde{\Psi}$ of (2.13) extends to a biholomorphic mapping of $U \times W$, from which it follows that

$$\Psi : E(\Gamma, \tau) \to A(\Gamma, j_\tau)$$

is an isomorphism of holomorphic bundles. Thus we have also, for instance,

$$\Omega_S^p \simeq A(\Gamma, j_{\Lambda^p \mathrm{Ad}^-}).$$

In this manner, holomorphic sections of bundles $E(\Gamma, \tau)$ become given as spaces of automorphic forms. One also uses (2.13) to construct local frames for $E(\Gamma, \tau)$.