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Autor: Zucker, Steven
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§2. VECTOR BUNDLES ON $\Gamma \backslash M$

Let Γ be a discrete subgroup of G which acts freely on the symmetric space M , and put $S = \Gamma \backslash M$. We will discuss two standard constructions of vector bundles on S .

The first type is the quotient by Γ of a homogeneous vector bundle on M . Specifically, let (τ, W) be a finite-dimensional representation of K . Then $E(\tau)$ is defined as the quotient of $G \times W$ by the following identification under the action of K :

$$(2.1) \quad (g, w) \sim (gk^{-1}, \tau(k)w) \quad \text{if} \quad k \in K.$$

$E(\tau)$ is naturally a C^∞ vector bundle on M , and the left action of G on M is covered by the obvious left action of G on $E(\tau)$. Thus, we may take the quotient by any Γ as above to obtain a bundle $E(\Gamma, \tau)$ on S . Alternatively, $E(\Gamma, \tau)$ is an associated vector bundle of the principal K -bundle $\Gamma \backslash G$. Note that if (τ, W) decomposes as a representation of K into

$$(\tau, W) = \bigoplus_{i=1}^l (\tau_i, W_i),$$

then one gets an induced decomposition

$$(2.2) \quad E(\Gamma, \tau) \simeq \bigoplus_{i=1}^l E(\Gamma, \tau_i).$$

We may identify sections of $E(\Gamma, \tau)$ as the Γ -invariant sections of $E(\tau)$, which in turn are given by mappings $\phi: G \rightarrow W$ which satisfy

$$(2.3) \quad \phi(\gamma g k^{-1}) = \tau(k) \phi(g) \quad \text{for all} \quad \gamma \in \Gamma, g \in G, k \in K.$$

An Hermitian metric can be placed on $E(\Gamma, \tau)$ by a choice of $\tau(K)$ -invariant inner product on W . (Such exist because K is compact.) The corresponding constant metric on $G \times W$ descends to $E(\Gamma, \tau)$, in view of (2.1).

(2.4) *Example.* Taking $\tau = \text{Ad}|_{\mathfrak{p}_\mathbb{C}}$, we have a natural isomorphism of $E(\tau)$ and the complexified tangent bundle to M , and we may take quotients by Γ .

The second type of vector bundle is the flat bundle associated to a finite-dimensional representation (ψ, V) of Γ . We let $\Phi(\psi)$ denote the quotient of $M \times V$ under the action of Γ :

$$(m, v) \sim (\gamma m, \psi(\gamma)v).$$

Sections of $\Phi(\psi)$ are given by functions $f: M \rightarrow V$ such that

$$(2.5) \quad f(\gamma x) = \psi(\gamma) f(x) \quad \text{if} \quad \gamma \in \Gamma, x \in M.$$

The local sections of $\Phi(\psi)$ determined by constant V -valued functions determine a flat structure on $\Phi(\psi)$, whose sheaf of locally constant sections will be denoted V .

The two constructions above are related by the elementary

(2.6) PROPOSITION. Let (ρ, V) be a representation of G (which then restricts to representations of K and Γ). Then the mapping

$$\tilde{\Xi}: G \times V \rightarrow G \times V,$$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of C^∞ vector bundles

$$\Xi: \Phi(\rho|_\Gamma) \rightarrow E(\Gamma, \rho|_K).$$

(2.7) Remark. Let (ρ, V) be a finite-dimensional representation of G , and (ψ, W) a finite-dimensional unitary representation of Γ . We note that by the standard ruse of replacing G by $G' = G \times U(W)$, where $U(W)$ denotes the unitary group of W , $V \otimes W$ becomes a representation space for G' , and so the bundle $\Phi(\rho|_\Gamma \otimes \psi)$ falls into the class of bundles covered by (2.6).

A natural metric on $\Phi(\rho|_\Gamma)$ is provided by the admissible inner product T (1.9). For $g \in G$, $v, w \in V$, let (at $gx_0 \in M$)

$$(2.8) \quad \langle v, w \rangle_{gx_0} = T(\rho(g^{-1})v, \rho(g^{-1})w).$$

Since K acts isometrically with respect to T , it follows that (2.8) is well-defined on $M \times V$; and it is evident that the action of Γ is isometric, so (2.8) descends to $\Phi(\rho|_\Gamma)$. T also determines a metric in $E(\Gamma, \rho|_K)$, and it is clear that the mapping Ξ of (2.6) is then an isometry of bundles.

Assume next that M is Hermitian. Then to every finite-dimensional holomorphic representation (σ, W) of Q is associated a $G_\mathbb{C}$ -equivariant holomorphic vector bundle $\tilde{E}(\sigma)$ on \tilde{M} , constructed as in (2.1). By restricting to M and taking the quotient by the action of Γ , we obtain the holomorphic vector bundle $\tilde{E}(\Gamma, \sigma)$ on S . Q -invariant subspaces of W determine holomorphic subbundles of $\tilde{E}(\Gamma, \sigma)$. Along the same lines as (2.6), we have:

(2.9) PROPOSITION. Let (ρ, V) be a representation of G (which then determines representations of Q and Γ). Then the mapping

$$\tilde{\Xi}: G_\mathbb{C} \times V \rightarrow G_\mathbb{C} \times V,$$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of holomorphic bundles

$$\Xi: \Phi(\rho|_{\Gamma}) \simeq \check{E}(\Gamma, \rho_c|_Q).$$

Every representation τ of K determines a holomorphic representation of $K_{\mathbb{C}}$, which then extends to a representation σ_{τ} of Q by setting σ_{τ} to be trivial on P^{-} , since K normalizes P^{-} . The C^{∞} isomorphism $E(\Gamma, \tau) \rightarrow \check{E}(\Gamma, \sigma_{\tau})$ imparts a holomorphic structure to $E(\Gamma, \tau)$; however, an isomorphism (2.2) need not be holomorphically compatible with (2.9).

(2.10) *Example.* Taking $\tau = \text{Ad}^{+} = \text{Ad } K|_{\mathfrak{p}^{+}}$ we obtain a holomorphic isomorphism

$$E(\tau) \simeq \Theta_M \quad (\text{holomorphic tangent bundle of } M),$$

and we may take quotients by Γ . Therefore, since the Killing form gives $(\mathfrak{p}^{+})^{*} \simeq \mathfrak{p}^{-}$ as a representation of K ,

$$E(\Gamma, \Lambda^p \text{Ad}^{-}) \simeq \Omega_S^p.$$

(Here and elsewhere, we identify a vector bundle with its locally free sheaf of germs of sections.)

There is a relation of the preceding to automorphic forms, coming from the following. Let W be a finite dimensional vector space over \mathbb{C} . Then an *automorphy factor* \mathscr{J} is a C^{∞} mapping

$$\mathscr{J}: G \times M \rightarrow GL(W)$$

which satisfies

- (2.11) i) $\mathscr{J}(g, x)$ is, for fixed g , a holomorphic mapping from M into $GL(W)$,
 ii) $\mathscr{J}(gh, x) = \mathscr{J}(g, hx) \mathscr{J}(h, x)$.

We observe that \mathscr{J} is then completely determined by the function $\mathscr{J}(g, x_0)$ on G . Given such a \mathscr{J} , one forms the *automorphic vector bundle* $A(\Gamma, \mathscr{J})$, a holomorphic bundle, by taking the quotient of $M \times W$ under the action of Γ :

$$(x, w) \sim (\gamma x, \mathscr{J}(\gamma, x) w) \quad \text{for all } \gamma \in \Gamma, x \in M, w \in W.$$

Sections of $A(\Gamma, \mathscr{J})$ are then given by functions $f: M \rightarrow W$ such that

$$(2.12) \quad f(\gamma x) = \mathscr{J}(\gamma, x) f(x) \quad \text{for all } \gamma \in \Gamma, x \in M;$$

these are called *automorphic forms*.

From an automorphy factor \mathcal{J} , one obtains a representation $\tau_{\mathcal{J}}$ of K by setting

$$\tau_{\mathcal{J}}(k) = \mathcal{J}(k, x_0),$$

because of (2.11, ii). We then have

(2.13) PROPOSITION. Let \mathcal{J} be an automorphy factor. Then there is a C^∞ isomorphism

$$\Psi: E(\Gamma, \tau_{\mathcal{J}}) \rightarrow A(\Gamma, \mathcal{J}),$$

induced by the mapping

$$\begin{aligned} \tilde{\Psi}: G \times W &\rightarrow G \times W \\ \tilde{\Psi}(g, w) &= (g, \mathcal{J}(g, x_0) w). \end{aligned}$$

(2.14) Remark. For a representation (ρ, V) of G ,

$$\mathcal{J}(g, x) = \rho(g)$$

defines an automorphy factor, for which (2.13) is a reformulation of (2.6).

Conversely, to the Lie group G is associated its *canonical automorphy factor* \mathcal{J} (see [7, p. 397]), which is a C^∞ mapping $\mathcal{J}: G \times M \rightarrow K_{\mathbb{C}}$ which satisfies the equations of (2.11); and $\mathcal{J}(g, x_0)$ is the $K_{\mathbb{C}}$ -component of g in $G \subset U = P^+ K_{\mathbb{C}} P^-$. Then each representation τ of K determines an automorphy factor

$$\mathcal{J}_\tau(g, x) = \tau(\mathcal{J}(g, x)).$$

In this case, the mapping $\tilde{\Psi}$ of (2.13) extends to a biholomorphic mapping of $U \times W$, from which it follows that

$$\Psi: E(\Gamma, \tau) \rightarrow A(\Gamma, \mathcal{J}_\tau)$$

is an isomorphism of holomorphic bundles. Thus we have also, for instance,

$$\Omega_S^p \simeq A(\Gamma, \mathcal{J}_{\wedge^p \text{Ad}^-}).$$

In this manner, holomorphic sections of bundles $E(\Gamma, \tau)$ become given as spaces of automorphic forms. One also uses (2.13) to construct local frames for $E(\Gamma, \tau)$.