

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 27 (1981)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE  
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**Kapitel:** §2. Vector bundles on  $\mathbb{M}$   
**DOI:** <https://doi.org/10.5169/seals-51751>

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§2. VECTOR BUNDLES ON  $\Gamma \backslash M$ 

Let  $\Gamma$  be a discrete subgroup of  $G$  which acts freely on the symmetric space  $M$ , and put  $S = \Gamma \backslash M$ . We will discuss two standard constructions of vector bundles on  $S$ .

The first type is the quotient by  $\Gamma$  of a homogeneous vector bundle on  $M$ . Specifically, let  $(\tau, W)$  be a finite-dimensional representation of  $K$ . Then  $E(\tau)$  is defined as the quotient of  $G \times W$  by the following identification under the action of  $K$ :

$$(2.1) \quad (g, w) \sim (gk^{-1}, \tau(k)w) \quad \text{if} \quad k \in K.$$

$E(\tau)$  is naturally a  $C^\infty$  vector bundle on  $M$ , and the left action of  $G$  on  $M$  is covered by the obvious left action of  $G$  on  $E(\tau)$ . Thus, we may take the quotient by any  $\Gamma$  as above to obtain a bundle  $E(\Gamma, \tau)$  on  $S$ . Alternatively,  $E(\Gamma, \tau)$  is an associated vector bundle of the principal  $K$ -bundle  $\Gamma \backslash G$ . Note that if  $(\tau, W)$  decomposes as a representation of  $K$  into

$$(\tau, W) = \bigoplus_{i=1}^l (\tau_i, W_i),$$

then one gets an induced decomposition

$$(2.2) \quad E(\Gamma, \tau) \simeq \bigoplus_{i=1}^l E(\Gamma, \tau_i).$$

We may identify sections of  $E(\Gamma, \tau)$  as the  $\Gamma$ -invariant sections of  $E(\tau)$ , which in turn are given by mappings  $\phi: G \rightarrow W$  which satisfy

$$(2.3) \quad \phi(\gamma g k^{-1}) = \tau(k) \phi(g) \quad \text{for all} \quad \gamma \in \Gamma, g \in G, k \in K.$$

An Hermitian metric can be placed on  $E(\Gamma, \tau)$  by a choice of  $\tau(K)$ -invariant inner product on  $W$ . (Such exist because  $K$  is compact.) The corresponding constant metric on  $G \times W$  descends to  $E(\Gamma, \tau)$ , in view of (2.1).

(2.4) *Example.* Taking  $\tau = \text{Ad}|_{\mathfrak{p}_\mathbb{C}}$ , we have a natural isomorphism of  $E(\tau)$  and the complexified tangent bundle to  $M$ , and we may take quotients by  $\Gamma$ .

The second type of vector bundle is the flat bundle associated to a finite-dimensional representation  $(\psi, V)$  of  $\Gamma$ . We let  $\Phi(\psi)$  denote the quotient of  $M \times V$  under the action of  $\Gamma$ :

$$(m, v) \sim (\gamma m, \psi(\gamma)v).$$

Sections of  $\Phi(\psi)$  are given by functions  $f: M \rightarrow V$  such that

$$(2.5) \quad f(\gamma x) = \psi(\gamma) f(x) \quad \text{if} \quad \gamma \in \Gamma, x \in M.$$

The local sections of  $\Phi(\psi)$  determined by constant  $V$ -valued functions determine a flat structure on  $\Phi(\psi)$ , whose sheaf of locally constant sections will be denoted  $V$ .

The two constructions above are related by the elementary

(2.6) PROPOSITION. *Let  $(\rho, V)$  be a representation of  $G$  (which then restricts to representations of  $K$  and  $\Gamma$ ). Then the mapping*

$$\tilde{\Xi}: G \times V \rightarrow G \times V,$$

defined by  $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1}v)$ , induces an isomorphism of  $C^\infty$  vector bundles

$$\Xi: \Phi(\rho|_\Gamma) \simeq E(\Gamma, \rho|_K).$$

(2.7) Remark. Let  $(\rho, V)$  be a finite-dimensional representation of  $G$ , and  $(\psi, W)$  a finite-dimensional unitary representation of  $\Gamma$ . We note that by the standard ruse of replacing  $G$  by  $G' = G \times U(W)$ , where  $U(W)$  denotes the unitary group of  $W$ ,  $V \otimes W$  becomes a representation space for  $G'$ , and so the bundle  $\Phi(\rho|_\Gamma \otimes \psi)$  falls into the class of bundles covered by (2.6).

A natural metric on  $\Phi(\rho|_\Gamma)$  is provided by the admissible inner product  $T$  (1.9). For  $g \in G$ ,  $v, w \in V$ , let (at  $gx_0 \in M$ )

$$(2.8) \quad \langle v, w \rangle_{gx_0} = T(\rho(g^{-1})v, \rho(g^{-1})w).$$

Since  $K$  acts isometrically with respect to  $T$ , it follows that (2.8) is well-defined on  $M \times V$ ; and it is evident that the action of  $\Gamma$  is isometric, so (2.8) descends to  $\Phi(\rho|_\Gamma)$ .  $T$  also determines a metric in  $E(\Gamma, \rho|_K)$ , and it is clear that the mapping  $\Xi$  of (2.6) is then an isometry of bundles.

Assume next that  $M$  is Hermitian. Then to every finite-dimensional holomorphic representation  $(\sigma, W)$  of  $Q$  is associated a  $G_C$ -equivariant holomorphic vector bundle  $\check{E}(\sigma)$  on  $\check{M}$ , constructed as in (2.1). By restricting to  $M$  and taking the quotient by the action of  $\Gamma$ , we obtain the holomorphic vector bundle  $\check{E}(\Gamma, \sigma)$  on  $S$ .  $Q$ -invariant subspaces of  $W$  determine holomorphic subbundles of  $\check{E}(\Gamma, \sigma)$ . Along the same lines as (2.6), we have:

(2.9) PROPOSITION. Let  $(\rho, V)$  be a representation of  $G$  (which then determines representations of  $Q$  and  $\Gamma$ ). Then the mapping

$$\tilde{\Xi}: G_C \times V \rightarrow G_C \times V,$$

defined by  $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1}v)$ , induces an isomorphism of holomorphic bundles

$$\Xi: \Phi(\rho|_{\Gamma}) \simeq \check{E}(\Gamma, \rho_c|_{\mathcal{Q}}).$$

Every representation  $\tau$  of  $K$  determines a holomorphic representation of  $K_{\mathbb{C}}$ , which then extends to a representation  $\sigma_{\tau}$  of  $Q$  by setting  $\sigma_{\tau}$  to be trivial on  $P^{-}$ , since  $K$  normalizes  $P^{-}$ . The  $C^{\infty}$  isomorphism  $E(\Gamma, \tau) \rightarrow \check{E}(\Gamma, \sigma_{\tau})$  imparts a holomorphic structure to  $E(\Gamma, \tau)$ ; however, an isomorphism (2.2) need not be holomorphically compatible with (2.9).

(2.10) *Example.* Taking  $\tau = \text{Ad}^{+} = \text{Ad } K|_{\mathfrak{p}^{+}}$  we obtain a holomorphic isomorphism

$$E(\tau) \simeq \Theta_M \quad (\text{holomorphic tangent bundle of } M),$$

and we may take quotients by  $\Gamma$ . Therefore, since the Killing form gives  $(\mathfrak{p}^{+})^{*} \simeq \mathfrak{p}^{-}$  as a representation of  $K$ ,

$$E(\Gamma, \Lambda^p \text{Ad}^{-}) \simeq \Omega_{\mathbb{S}}^p.$$

(Here and elsewhere, we identify a vector bundle with its locally free sheaf of germs of sections.)

There is a relation of the preceding to automorphic forms, coming from the following. Let  $W$  be a finite dimensional vector space over  $\mathbb{C}$ . Then an *automorphy factor*  $\mathcal{f}$  is a  $C^{\infty}$  mapping

$$\mathcal{f}: G \times M \rightarrow GL(W)$$

which satisfies

- (2.11) i)  $\mathcal{f}(g, x)$  is, for fixed  $g$ , a holomorphic mapping from  $M$  into  $GL(W)$ ,  
 ii)  $\mathcal{f}(gh, x) = \mathcal{f}(g, hx) \mathcal{f}(h, x)$ .

We observe that  $\mathcal{f}$  is then completely determined by the function  $\mathcal{f}(g, x_0)$  on  $G$ . Given such a  $\mathcal{f}$ , one forms the *automorphic vector bundle*  $A(\Gamma, \mathcal{f})$ , a holomorphic bundle, by taking the quotient of  $M \times W$  under the action of  $\Gamma$ :

$$(x, w) \sim (\gamma x, \mathcal{f}(\gamma, x)w) \quad \text{for all } \gamma \in \Gamma, x \in M, w \in W.$$

Sections of  $A(\Gamma, \mathcal{f})$  are then given by functions  $f: M \rightarrow W$  such that

$$(2.12) \quad f(\gamma x) = \mathcal{f}(\gamma, x) f(x) \quad \text{for all } \gamma \in \Gamma, x \in M;$$

these are called *automorphic forms*.

From an automorphy factor  $f$ , one obtains a representation  $\tau_f$  of  $K$  by setting

$$\tau_f(k) = f(k, x_0),$$

because of (2.11, ii). We then have

(2.13) PROPOSITION. *Let  $f$  be an automorphy factor. Then there is a  $C^\infty$  isomorphism*

$$\Psi: E(\Gamma, \tau_f) \rightarrow A(\Gamma, f),$$

induced by the mapping

$$\begin{aligned} \tilde{\Psi}: G \times W &\rightarrow G \times W \\ \tilde{\Psi}(g, w) &= (g, f(g, x_0)w). \end{aligned}$$

(2.14) Remark. For a representation  $(\rho, V)$  of  $G$ ,

$$f(g, x) = \rho(g)$$

defines an automorphy factor, for which (2.13) is a reformulation of (2.6).

Conversely, to the Lie group  $G$  is associated its *canonical automorphy factor*  $\mathcal{J}$  (see [7, p. 397]), which is a  $C^\infty$  mapping  $\mathcal{J}: G \times M \rightarrow K_C$  which satisfies the equations of (2.11); and  $\mathcal{J}(g, x_0)$  is the  $K_C$ -component of  $g$  in  $G \subset U = P^+ K_C P^-$ . Then each representation  $\tau$  of  $K$  determines an automorphy factor

$$f_\tau(g, x) = \tau(\mathcal{J}(g, x)).$$

In this case, the mapping  $\tilde{\Psi}$  of (2.13) extends to a biholomorphic mapping of  $U \times W$ , from which it follows that

$$\Psi: E(\Gamma, \tau) \rightarrow A(\Gamma, f_\tau)$$

is an isomorphism of holomorphic bundles. Thus we have also, for instance,

$$\Omega_S^p \simeq A(\Gamma, f_{\wedge^p \text{Ad}^-}).$$

In this manner, holomorphic sections of bundles  $E(\Gamma, \tau)$  become given as spaces of automorphic forms. One also uses (2.13) to construct local frames for  $E(\Gamma, \tau)$ .