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§2. Vector bundles on $\Gamma \setminus M$

Let Γ be a discrete subgroup of G which acts freely on the symmetric space M, and put $S = \Gamma \setminus M$. We will discuss two standard constructions of vector bundles on S.

The first type is the quotient by Γ of a homogeneous vector bundle on M. Specifically, let (τ, W) be a finite-dimensional representation of K. Then $E(\tau)$ is defined as the quotient of $G \times W$ by the following identification under the action of K:

(2.1)
$$(g, w) \sim (gk^{-1}, \tau(k) w) \quad \text{if} \quad k \in K .$$

 $E(\tau)$ is naturally a C^{∞} vector bundle on M, and the left action of G on M is covered by the obvious left action of G on $E(\tau)$. Thus, we may take the quotient by any Γ as above to obtain a bundle $E(\Gamma, \tau)$ on S. Alternatively, $E(\Gamma, \tau)$ is an associated vector bundle of the principal K-bundle $\Gamma \setminus G$. Note that if (τ, W) decomposes as a representation of K into

$$(\tau, W) = \bigoplus_{i=1}^{l} (\tau_i, W_i),$$

then one gets an induced decomposition

(2.2) $E(\Gamma, \tau) \simeq \bigoplus_{i=1}^{l} E(\Gamma, \tau_i).$

We may identify sections of $E(\Gamma, \tau)$ as the Γ -invariant sections of $E(\tau)$, which in turn are given by mappings $\phi: G \to W$ which satisfy

(2.3) $\phi(\gamma g k^{-1}) = \tau(k) \phi(g)$ for all $\gamma \in \Gamma, g \in G, k \in K$.

An Hermitian metric can be placed on $E(\Gamma, \tau)$ by a choice of $\tau(K)$ -invariant inner product on W. (Such exist because K is compact.) The corresponding constant metric on $G \times W$ descends to $E(\Gamma, \tau)$, in view of (2.1).

(2.4) *Example.* Taking $\tau = \operatorname{Ad} |_{\mathfrak{p}_{\mathbf{C}}}$, we have a natural isomorphism of $E(\tau)$ and the complexified tangent bundle to M, and we may take quotients by Γ .

The second type of vector bundle is the flat bundle associated to a finitedimensional representation (ψ, V) of Γ . We let $\Phi(\psi)$ denote the quotient of $M \times V$ under the action of Γ :

$$(m, v) \sim (\gamma m, \psi(\gamma) v).$$

Sections of $\Phi(\psi)$ are given by functions $f: M \to V$ such that

(2.5)
$$f(\gamma x) = \psi(\gamma) f(x)$$
 if $\gamma \in \Gamma, x \in M$.

The local sections of $\Phi(\psi)$ determined by constant V-valued functions determine a flat structure on $\Phi(\psi)$, whose sheaf of locally constant sections will be denoted V.

The two constructions above are related by the elementary

(2.6) PROPOSITION. Let (ρ, V) be a representation of G (which then restricts to representations of K and Γ). Then the mapping

 $\tilde{\Xi}: G \times V \to G \times V \,,$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of C^{∞} vector bundles

$$\Xi: \Phi(\rho|_{\Gamma}) \cong E(\Gamma, \rho|_{K}).$$

(2.7) Remark. Let (ρ, V) be a finite-dimensional representation of G, and (ψ, W) a finite-dimensional unitary representation of Γ . We note that by the standard ruse of replacing G by $G' = G \times U(W)$, where U(W) denotes the unitary group of $W, V \otimes W$ becomes a representation space for G', and so the bundle $\Phi(\rho|_{\Gamma} \otimes \psi)$ falls into the class of bundles covered by (2.6).

A natural metric on $\Phi(\rho|_{\Gamma})$ is provided by the admissible inner product T (1.9). For $g \in G$, $v, w \in V$, let (at $gx_0 \in M$)

(2.8) $\langle v, w \rangle_{gx_0} = T\left(\rho\left(g^{-1}\right)v, \rho\left(g^{-1}\right)w\right).$

Since K acts isometrically with respect to T, it follows that (2.8) is well-defined on $M \times V$; and it is evident that the action of Γ is isometric, so (2.8) descends to $\Phi(\rho|_{\Gamma})$. T also determines a metric in $E(\Gamma, \rho|_{K})$, and it is clear that the mapping Ξ of (2.6) is then an isometry of bundles.

Assume next that M is Hermitian. Then to every finite-dimensional holomorphic representation (σ, W) of Q is associated a G_c -equivariant holomorphic vector bundle $\check{E}(\sigma)$ on \check{M} , constructed as in (2.1). By restricting to M and taking the quotient by the action of Γ , we obtain the holomorphic vector bundle $\check{E}(\Gamma, \sigma)$ on S. Q-invariant subspaces of W determine holomorphic subbundles of $\check{E}(\Gamma, \sigma)$. Along the same lines as (2.6), we have:

(2.9) PROPOSITION. Let (ρ, V) be a representation of G (which then determines representations of Q and Γ). Then the mapping

$$\tilde{\Xi}: G_{\mathbf{C}} \times V \to G_{\mathbf{C}} \times V,$$

defined by $\tilde{\Xi}(g, v) = (g, \rho(g)^{-1} v)$, induces an isomorphism of holomorphic bundles

$$\Xi: \Phi(\rho|_{\Gamma}) \cong \check{E}(\Gamma, \rho_{\mathbf{c}}|_{\boldsymbol{o}}).$$

Every representation τ of K determines a holomorphic representation of K_c , which then extends to a representation σ_{τ} of Q by setting σ_{τ} to be trivial on P^- , since K normalizes P^- . The C^{∞} isomorphism $E(\Gamma, \tau) \rightarrow \check{E}(\Gamma, \sigma_{\tau})$ imparts a holomorphic structure to $E(\Gamma, \tau)$; however, an isomorphism (2.2) need not be holomorphically compatible with (2.9).

(2.10) *Example.* Taking $\tau = Ad^+ = Ad K |_{p^+}$ we obtain a holomorphic isomorphism

 $E(\tau) \simeq \Theta_M$ (holomorphic tangent bundle of M),

and we may take quotients by Γ . Therefore, since the Killing form gives $(p^+)^* \simeq p^-$ as a representation of K,

$$E(\Gamma, \Lambda^p \mathrm{Ad}^-) \simeq \Omega^p_{\mathrm{S}}.$$

(Here and elsewhere, we identify a vector bundle with its locally free sheaf of germs of sections.)

There is a relation of the preceding to automorphic forms, coming from the following. Let W be a finite dimensional vector space over \mathbf{C} . Then an automorphy factor \mathbf{j} is a C^{∞} mapping

$$\mathcal{J}:G\times M\to GL\left(W\right)$$

which satisfies

(2.11) i)
$$f(g, x)$$
 is, for fixed g , a holomorphic mapping from M into $GL(W)$,
ii) $f(gh, x) = f(g, hx) f(h, x)$.

We observe that f is then completely determined by the function $f(g, x_0)$ on G. Given such a f, one forms the *automorphic vector bundle* $A(\Gamma, f)$, a holomorphic bundle, by taking the quotient of $M \times W$ under the action of Γ :

 $(x, w) \sim (\gamma x, \gamma(\gamma, x) w)$ for all $\gamma \in \Gamma, x \in M, w \in W$.

Sections of $A(\Gamma, \mathcal{J})$ are then given by functions $f: M \to W$ such that

(2.12)
$$f(\gamma x) = \mathcal{J}(\gamma, x) f(x)$$
 for all $\gamma \in \Gamma, x \in M$;

these are called automorphic forms.

From an automorphy factor f, one obtains a representation τ_f of K by setting

$$\mathbf{t}_{\mathbf{j}}(k) = \mathbf{j}(k, x_0),$$

because of (2.11, ii). We then have

(2.13) **PROPOSITION.** Let \mathcal{J} be an automorphy factor. Then there is a C^{∞} isomorphism

$$\Psi: E\left(\Gamma, \tau_{j}\right) \to A\left(\Gamma_{j}\right),$$

induced by the mapping

$$\begin{split} \widetilde{\Psi} &: G \times W \to G \times W \\ \widetilde{\Psi} &(g, w) = \left(g, \swarrow & (g, x_0) w \right). \end{split}$$

(2.14) Remark. For a representation (ρ, V) of G,

 $\mathcal{J}(g, x) = \rho(g)$

defines an automorphy factor, for which (2.13) is a reformulation of (2.6).

Conversely, to the Lie group G is associated its canonical automorphy factor \mathscr{J} (see [7, p. 397]), which is a C^{∞} mapping $\mathscr{J}: G \times M \to K_{\mathbf{C}}$ which satisfies the equations of (2.11); and $\mathscr{J}(g, x_0)$ is the $K_{\mathbf{C}}$ -component of g in $G \subset U$ = $P^+ K_{\mathbf{C}} P^-$. Then each representation τ of K determines an automorphy factor

$$\mathcal{J}_{\tau}(g, x) = \tau \left(\mathcal{J}(g, x) \right).$$

In this case, the mapping Ψ of (2.13) extends to a biholomorphic mapping of $U \times W$, from which it follows that

$$\Psi \colon E(\Gamma, \tau) \to A(\Gamma, f_{\tau})$$

is an isomorphism of holomorphic bundles. Thus we have also, for instance,

$$\Omega^p_S \simeq A\left(\Gamma, \mathcal{J}_{\Lambda^p \mathrm{Ad}}^{-}\right).$$

In this manner, holomorphic sections of bundles $E(\Gamma, \tau)$ become given as spaces of automorphic forms. One also uses (2.13) to construct local frames for $E(\Gamma, \tau)$.