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## §1. PRELIMINARIES

Let  $G$  be a connected real semi-simple Lie group with finite center,  $K$  a maximal compact subgroup of  $G$ , and let  $\mathfrak{g} \supset \mathfrak{k}$  be the corresponding Lie algebras. For any sub-algebra  $\mathfrak{a} \subset \mathfrak{g}$ , we put

$$\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}.$$

If  $B$  denotes the Killing form of  $\mathfrak{g}$ ,  $B$  is negative-definite on  $\mathfrak{k}$ , and we let  $\mathfrak{p}$  denote the orthogonal complement under  $B$  of  $\mathfrak{k}$  in  $\mathfrak{g}$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a so-called Cartan decomposition of  $\mathfrak{g}$ , and  $B$  is positive-definite on  $\mathfrak{p}$ .

Let  $M = G/K$ , the corresponding symmetric space. As  $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ ,  $B$  defines an  $(\text{Ad } K)$ -invariant inner product on  $\mathfrak{p}$ ; and since we may identify  $\mathfrak{p}$  naturally as the tangent space to  $M$  at the identity coset  $x_0 = K$ ,  $B$  determines a unique Riemannian metric on  $M$  which is invariant under the canonical left  $G$ -action.

Assume initially that  $M$  is an *irreducible* symmetric space. Then, if one wishes,  $G$  can be taken to be a non-compact almost simple group (i.e.,  $\mathfrak{g}$  is a simple Lie algebra). In that case, the space  $M$  admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when  $\mathfrak{k}$  has a non-trivial center  $\mathfrak{z}$ . In this case,  $\dim \mathfrak{z} = 1$ , and  $Z = \exp \mathfrak{z}$  is the identity component of the center of  $K$ . Let  $G^{\text{ad}}$  denote the adjoint group of  $G$  (i.e., the automorphism group of  $M$ ) and let  $K^{\text{ad}}, Z^{\text{ad}}$  be the corresponding subgroups of  $G^{\text{ad}}$ . A choice of  $z_0 \in Z^{\text{ad}}$  of order 4 (for which  $\text{Ad}(z_0^2)$  is a Cartan involution of  $\mathfrak{g}$ ) determines an almost-complex structure on  $\mathfrak{p}$ :

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with

$$(1.1) \quad \begin{cases} \mathfrak{p}^+ = \{X \in \mathfrak{p}_{\mathbb{C}} : \text{Ad}(z_0) X = iX\} \\ \mathfrak{p}^- = \{X \in \mathfrak{p}_{\mathbb{C}} : \text{Ad}(z_0) X = -iX\} \end{cases}$$

This determines, via left-translation under  $G$ , a Kählerian complex structure on  $M$ , such that the action of  $G$  is by holomorphic isometries.

For purposes of numeration, we define  $\mu = \mu(G)$  to be the degree of the covering map  $Z \rightarrow Z^{\text{ad}}$ . It has the following properties:

- (1.2)    i) if  $G$  is of adjoint type,  $\mu = 1$  (cf. [16, (1.17B)]),  
           ii) if  $G' \rightarrow G$  is a finite covering, then  $\mu(G)$  divides  $\mu(G')$ ,  
           iii) if  $G = SU(n, 1)$ , then  $\mu = n + 1$ .

Let  $\rho$  be an irreducible representation of  $G$  on the finite-dimensional complex vector space  $V$ . We will say that  $(\rho, V)$  is a *real* representation if there is a  $G$ -invariant  $\mathbf{R}$ -subspace  $V_{\mathbf{R}}$  of  $V$  with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that  $G$  acts on  $V$  by extension of scalars. Under the subgroup  $Z$ , the representation necessarily splits into one-dimensional  $Z$ -invariant summands, on each of which  $Z$  acts by a character. We pick an isomorphism

$$(1.3) \quad \phi: Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of  $Z$  is free cyclic, with elements

$$\chi_n: Z \rightarrow S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

$$(1.4) \quad V \langle n \rangle = \{v \in V : \rho(z)v = \chi_n(z)v \quad \text{if} \quad z \in Z\},$$

so that

$$(1.5) \quad V = \bigoplus_{n \in \mathbf{Z}} V \langle n \rangle.$$

Each  $V \langle n \rangle$  is invariant under  $K$ . If  $(\tau, W)$  is a representation of  $K$ , then we define  $W \langle n \rangle$  as in (1.4); if  $W$  is irreducible, then  $W = W \langle n \rangle$  for some  $n$ ,

i.e.,  $Z$  acts by a single character. We will assume to have chosen the isomorphism (1.3) so that  $Z$  acts on  $\mathfrak{p}^+$  by a "positive" character.

(1.6) *Example.* Assuming that  $G$  is almost simple, we take  $V_{\mathbf{R}} = \mathfrak{g}$ , and  $\rho = \text{Ad}$ , the adjoint representation of  $G$ . Then  $\mathfrak{p}^+ = V\langle\mu\rangle$ ,  $\mathfrak{k}_{\mathbf{C}} = V\langle 0\rangle$ , and  $\mathfrak{p}^- = V\langle -\mu\rangle$ .

For an irreducible (finite-dimensional) representation of  $G$ , we also use  $\rho$  to denote the induced action of  $\mathfrak{g}$  on  $V$ . Because of the above description (1.6) of  $\text{Ad}$ , it is easy to see that the following hold:

- (1.7) i)  $\rho(\mathfrak{p}^+) V\langle n\rangle \subset V\langle n+\mu\rangle$ ,  $\rho(\mathfrak{k}) V\langle n\rangle \subset V\langle n\rangle$ ,  
 $\rho(\mathfrak{p}^-) V\langle n\rangle \subset V\langle n-\mu\rangle$ .  
 ii)  $\{n: V\langle n\rangle \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, \dots, \lambda - m\mu\}$   
 for some integers  $\lambda \geq 0, m \geq \mu^{-1}\lambda$ .  
 iii) If  $V$  is real, then for all  $n$ ,  

$$V\langle -n\rangle = \overline{V\langle n\rangle} \quad (\text{complex conjugate})$$
  
 and thus  $m\mu = 2\lambda$ .

((1.7 i) includes, in particular, the standard fact that  $\mathfrak{p}^+$  and  $\mathfrak{p}^-$  are Abelian Lie subalgebras of  $\mathfrak{g}_{\mathbf{C}}$ .)

For the general case, write

$$(1.8) \quad G = (\prod_{j=1}^l G_j) \times H,$$

where each  $G_j$  is almost simple and of non-compact Hermitian type, and  $H$  is compact.<sup>1)</sup> Let

$$K = (\prod_{j=1}^l K_j) \times H;$$

$Z = \prod_{j=1}^l Z_j$ ; and  $Z^{\text{ad}} = \prod_{j=1}^l Z_j^{\text{ad}}$ , a product of circles. Let  $\Delta^{\text{ad}}$  be the diagonal of  $Z^{\text{ad}}$ , and  $\Delta$  the inverse image of  $\Delta^{\text{ad}}$  in  $Z$ . One may proceed as before, if we replace  $Z$  by  $\Delta$ . Alternatively, every irreducible representation  $(\rho, V)$  of  $G$  decomposes as a tensor product

$$(\otimes_{j=1}^l (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of  $\Delta$ , the decomposition (1.5) of  $V$  is the tensor product of the

<sup>1)</sup> We allow compact factors because of (2.7).

corresponding decompositions of each  $V_j$  into character spaces under  $Z_j$ , tensored with the "trivial" factor  $W$ .

On  $V$  there exists a positive-definite Hermitian form (the *admissible inner product*)  $T(v, w)$  (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$(1.9) \quad \begin{aligned} \text{i) } & T(\rho(k)v, \rho(k)w) = T(v, w) \quad \text{if } k \in K \\ \text{ii) } & T(\rho(X)v, w) = T(v, \rho(X)w) \quad \text{if } X \in \mathfrak{p}. \end{aligned}$$

This follows from the fact that  $\mathfrak{k} \oplus i\mathfrak{p}$  is a compact Lie algebra. If  $V$  is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on  $V_{\mathbf{R}}$ .

Let  $I$  denote the intersection of the kernels of all finite-dimensional representations of  $G$ . Then  $I$  is a central subgroup, and  $G/I$  admits the structure of a real (affine) algebraic group. Since we are interested in  $G$  only for its finite-dimensional representations and the symmetric space  $M$ , we may replace  $G$  by  $G/I$  and assume that  $G$  is an algebraic group. To get all of the representations of  $\mathfrak{g}$ , it is convenient in the abstract to replace  $G$  by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of  $G$ ); thus, *we may and do assume that  $G$  is algebraically simply connected*. We remark that by (1.2), the number  $\mu(G)$  can be arbitrarily large, even under this restriction.

Let, then,  $G_{\mathbf{C}}$  denote the set of complex points of  $G$ . It is a simply-connected complex Lie group with Lie algebra  $\mathfrak{g}_{\mathbf{C}}$ . Let  $K_{\mathbf{C}}$  denote the connected subgroup of  $G_{\mathbf{C}}$  with Lie algebra  $\mathfrak{k}_{\mathbf{C}}$ . By general theory (see [17, XVII.5]),  $K_{\mathbf{C}}$  is the universal complexification of  $K$ , and so, by definition, every representation of  $K$  extends to a holomorphic representation of  $K_{\mathbf{C}}$ .

Assume that  $M$  is Hermitian, and let  $P^+$  (resp.  $P^-$ ) denote the subgroup of  $G_{\mathbf{C}}$  corresponding to the subalgebra  $\mathfrak{p}^+$  (resp.  $\mathfrak{p}^-$ ) of  $\mathfrak{g}_{\mathbf{C}}$ . Then  $P^+K_{\mathbf{C}}P^-$  is an open subset of  $G_{\mathbf{C}}$  which contains  $G$  (see [4, p. 317]). Moreover,  $G \cap K_{\mathbf{C}}P^- = K$  (see [4, p. 318]), so the mapping of  $G \rightarrow G_{\mathbf{C}}$  induces a holomorphic embedding

$$(1.10) \quad M \rightarrow \check{M} = G_{\mathbf{C}}/Q;$$

as  $Q = K_{\mathbf{C}}P^-$  is a parabolic subgroup of  $G_{\mathbf{C}}$ ,  $\check{M}$  is compact and is known as the *compact dual* of  $M$ .