

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE
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Kapitel: §1. Preliminaries
DOI: <https://doi.org/10.5169/seals-51751>

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TABLE OF CONTENTS

1. Preliminaries	246
2. Vector bundles on $\Gamma \backslash M$	250
3. The cohomology groups $H^n(\Gamma; \rho, V)$	254
4. The variation of Hodge structure associated to (ρ, V)	261
5. Hodge theory for $H^n(\Gamma; \rho, V)$, from the variation of Hodge structure	264
REFERENCES	276

§1. PRELIMINARIES

Let G be a connected real semi-simple Lie group with finite center, K a maximal compact subgroup of G , and let $\mathfrak{g} \supset \mathfrak{k}$ be the corresponding Lie algebras. For any sub-algebra $\mathfrak{a} \subset \mathfrak{g}$, we put

$$\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}.$$

If B denotes the Killing form of \mathfrak{g} , B is negative-definite on \mathfrak{k} , and we let \mathfrak{p} denote the orthogonal complement under B of \mathfrak{k} in \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a so-called Cartan decomposition of \mathfrak{g} , and B is positive-definite on \mathfrak{p} .

Let $M = G/K$, the corresponding symmetric space. As $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, B defines an $(\text{Ad } K)$ -invariant inner product on \mathfrak{p} ; and since we may identify \mathfrak{p} naturally as the tangent space to M at the identity coset $x_0 = K$, B determines a unique Riemannian metric on M which is invariant under the canonical left G -action.

Assume initially that M is an *irreducible* symmetric space. Then, if one wishes, G can be taken to be a non-compact almost simple group (i.e., \mathfrak{g} is a simple Lie algebra). In that case, the space M admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when \mathfrak{k} has a non-trivial center \mathfrak{z} . In this case, $\dim \mathfrak{z} = 1$, and $Z = \exp \mathfrak{z}$ is the identity component of the center of K . Let G^{ad} denote the adjoint group of G (i.e., the automorphism group of M) and let $K^{\text{ad}}, Z^{\text{ad}}$ be the corresponding subgroups of G^{ad} . A choice of $z_0 \in Z^{\text{ad}}$ of order 4 (for which $\text{Ad}(z_0^2)$ is a Cartan involution of \mathfrak{g}) determines an almost-complex structure on \mathfrak{p} :

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with

$$(1.1) \quad \begin{cases} \mathfrak{p}^+ = \{X \in \mathfrak{p}_{\mathbb{C}} : \text{Ad}(z_0) X = iX\} \\ \mathfrak{p}^- = \{X \in \mathfrak{p}_{\mathbb{C}} : \text{Ad}(z_0) X = -iX\} \end{cases}$$

This determines, via left-translation under G , a Kählerian complex structure on M , such that the action of G is by holomorphic isometries.

For purposes of numeration, we define $\mu = \mu(G)$ to be the degree of the covering map $Z \rightarrow Z^{\text{ad}}$. It has the following properties:

- (1.2) i) if G is of adjoint type, $\mu = 1$ (cf. [16, (1.17B)]),
 ii) if $G' \rightarrow G$ is a finite covering, then $\mu(G)$ divides $\mu(G')$,
 iii) if $G = SU(n, 1)$, then $\mu = n + 1$.

Let ρ be an irreducible representation of G on the finite-dimensional complex vector space V . We will say that (ρ, V) is a *real* representation if there is a G -invariant \mathbf{R} -subspace $V_{\mathbf{R}}$ of V with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that G acts on V by extension of scalars. Under the subgroup Z , the representation necessarily splits into one-dimensional Z -invariant summands, on each of which Z acts by a character. We pick an isomorphism

$$(1.3) \quad \phi: Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of Z is free cyclic, with elements

$$\chi_n: Z \rightarrow S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

$$(1.4) \quad V \langle n \rangle = \{v \in V : \rho(z)v = \chi_n(z)v \quad \text{if} \quad z \in Z\},$$

so that

$$(1.5) \quad V = \bigoplus_{n \in \mathbf{Z}} V \langle n \rangle.$$

Each $V \langle n \rangle$ is invariant under K . If (τ, W) is a representation of K , then we define $W \langle n \rangle$ as in (1.4); if W is irreducible, then $W = W \langle n \rangle$ for some n ,

i.e., Z acts by a single character. We will assume to have chosen the isomorphism (1.3) so that Z acts on \mathfrak{p}^+ by a "positive" character.

(1.6) *Example.* Assuming that G is almost simple, we take $V_{\mathbf{R}} = \mathfrak{g}$, and $\rho = \text{Ad}$, the adjoint representation of G . Then $\mathfrak{p}^+ = V\langle\mu\rangle$, $\mathfrak{k}_{\mathbf{C}} = V\langle 0\rangle$, and $\mathfrak{p}^- = V\langle-\mu\rangle$.

For an irreducible (finite-dimensional) representation of G , we also use ρ to denote the induced action of \mathfrak{g} on V . Because of the above description (1.6) of Ad , it is easy to see that the following hold:

- (1.7) i) $\rho(\mathfrak{p}^+)V\langle n\rangle \subset V\langle n+\mu\rangle$, $\rho(\mathfrak{k})V\langle n\rangle \subset V\langle n\rangle$,
 $\rho(\mathfrak{p}^-)V\langle n\rangle \subset V\langle n-\mu\rangle$.
 ii) $\{n: V\langle n\rangle \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, \dots, \lambda - m\mu\}$
 for some integers $\lambda \geq 0, m \geq \mu^{-1}\lambda$.
 iii) If V is real, then for all n ,
 $V\langle -n\rangle = \overline{V\langle n\rangle}$ (complex conjugate)
 and thus $m\mu = 2\lambda$.

((1.7 i) includes, in particular, the standard fact that \mathfrak{p}^+ and \mathfrak{p}^- are Abelian Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$.)

For the general case, write

$$(1.8) \quad G = (\prod_{j=1}^l G_j) \times H,$$

where each G_j is almost simple and of non-compact Hermitian type, and H is compact.¹⁾ Let

$$K = (\prod_{j=1}^l K_j) \times H;$$

$Z = \prod_{j=1}^l Z_j$; and $Z^{\text{ad}} = \prod_{j=1}^l Z_j^{\text{ad}}$, a product of circles. Let Δ^{ad} be the diagonal of Z^{ad} , and Δ the inverse image of Δ^{ad} in Z . One may proceed as before, if we replace Z by Δ . Alternatively, every irreducible representation (ρ, V) of G decomposes as a tensor product

$$(\otimes_{j=1}^l (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of Δ , the decomposition (1.5) of V is the tensor product of the

¹⁾ We allow compact factors because of (2.7).

corresponding decompositions of each V_j into character spaces under Z_j , tensored with the “trivial” factor W .

On V there exists a positive-definite Hermitian form (the *admissible inner product*) $T(v, w)$ (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$(1.9) \quad \begin{aligned} \text{i) } & T(\rho(k)v, \rho(k)w) = T(v, w) \quad \text{if } k \in K \\ \text{ii) } & T(\rho(X)v, w) = T(v, \rho(X)w) \quad \text{if } X \in \mathfrak{p}. \end{aligned}$$

This follows from the fact that $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie algebra. If V is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on $V_{\mathbf{R}}$.

Let I denote the intersection of the kernels of all finite-dimensional representations of G . Then I is a central subgroup, and G/I admits the structure of a real (affine) algebraic group. Since we are interested in G only for its finite-dimensional representations and the symmetric space M , we may replace G by G/I and assume that G is an algebraic group. To get all of the representations of \mathfrak{g} , it is convenient in the abstract to replace G by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of G); thus, *we may and do assume that G is algebraically simply connected*. We remark that by (1.2), the number $\mu(G)$ can be arbitrarily large, even under this restriction.

Let, then, $G_{\mathbf{C}}$ denote the set of complex points of G . It is a simply-connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let $K_{\mathbf{C}}$ denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. By general theory (see [17, XVII.5]), $K_{\mathbf{C}}$ is the universal complexification of K , and so, by definition, every representation of K extends to a holomorphic representation of $K_{\mathbf{C}}$.

Assume that M is Hermitian, and let P^+ (resp. P^-) denote the subgroup of $G_{\mathbf{C}}$ corresponding to the subalgebra \mathfrak{p}^+ (resp. \mathfrak{p}^-) of $\mathfrak{g}_{\mathbf{C}}$. Then $P^+K_{\mathbf{C}}P^-$ is an open subset of $G_{\mathbf{C}}$ which contains G (see [4, p. 317]). Moreover, $G \cap K_{\mathbf{C}}P^- = K$ (see [4, p. 318]), so the mapping of $G \rightarrow G_{\mathbf{C}}$ induces a holomorphic embedding

$$(1.10) \quad M \rightarrow \check{M} = G_{\mathbf{C}}/Q;$$

as $Q = K_{\mathbf{C}}P^-$ is a parabolic subgroup of $G_{\mathbf{C}}$, \check{M} is compact and is known as the *compact dual* of M .