

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 27 (1981)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE  
**Autor:** Zucker, Steven  
**Kapitel:** §1. Preliminaries  
**DOI:** <https://doi.org/10.5169/seals-51751>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 22.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## TABLE OF CONTENTS

1. Preliminaries . . . . .	246
2. Vector bundles on $\Gamma \backslash M$ . . . . .	250
3. The cohomology groups $H^n(\Gamma; \rho, V)$ . . . . .	254
4. The variation of Hodge structure associated to $(\rho, V)$ . . . . .	261
5. Hodge theory for $H^n(\Gamma; \rho, V)$ , from the variation of Hodge structure . . . . .	264
REFERENCES . . . . .	276

## §1. PRELIMINARIES

Let  $G$  be a connected real semi-simple Lie group with finite center,  $K$  a maximal compact subgroup of  $G$ , and let  $\mathfrak{g} \supset \mathfrak{k}$  be the corresponding Lie algebras. For any sub-algebra  $\mathfrak{a} \subset \mathfrak{g}$ , we put

$$\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}.$$

If  $B$  denotes the Killing form of  $\mathfrak{g}$ ,  $B$  is negative-definite on  $\mathfrak{k}$ , and we let  $\mathfrak{p}$  denote the orthogonal complement under  $B$  of  $\mathfrak{k}$  in  $\mathfrak{g}$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a so-called Cartan decomposition of  $\mathfrak{g}$ , and  $B$  is positive-definite on  $\mathfrak{p}$ .

Let  $M = G/K$ , the corresponding symmetric space. As  $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ ,  $B$  defines an  $(\text{Ad } K)$ -invariant inner product on  $\mathfrak{p}$ ; and since we may identify  $\mathfrak{p}$  naturally as the tangent space to  $M$  at the identity coset  $x_0 = K$ ,  $B$  determines a unique Riemannian metric on  $M$  which is invariant under the canonical left  $G$ -action.

Assume initially that  $M$  is an *irreducible* symmetric space. Then, if one wishes,  $G$  can be taken to be a non-compact almost simple group (i.e.,  $\mathfrak{g}$  is a simple Lie algebra). In that case, the space  $M$  admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when  $\mathfrak{k}$  has a non-trivial center  $\mathfrak{z}$ . In this case,  $\dim \mathfrak{z} = 1$ , and  $Z = \exp \mathfrak{z}$  is the identity component of the center of  $K$ . Let  $G^{\text{ad}}$  denote the adjoint group of  $G$  (i.e., the automorphism group of  $M$ ) and let  $K^{\text{ad}}, Z^{\text{ad}}$  be the corresponding subgroups of  $G^{\text{ad}}$ . A choice of  $z_0 \in Z^{\text{ad}}$  of order 4 (for which  $\text{Ad}(z_0^2)$  is a Cartan involution of  $\mathfrak{g}$ ) determines an almost-complex structure on  $\mathfrak{p}$ :

$$\mathfrak{p}_c = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with

$$(1.1) \quad \begin{cases} \mathfrak{p}^+ = \{X \in \mathfrak{p}_c : \text{Ad}(z_0) X = iX\} \\ \mathfrak{p}^- = \{X \in \mathfrak{p}_c : \text{Ad}(z_0) X = -iX\} \end{cases}$$

This determines, via left-translation under  $G$ , a Kählerian complex structure on  $M$ , such that the action of  $G$  is by holomorphic isometries.

For purposes of numeration, we define  $\mu = \mu(G)$  to be the degree of the covering map  $Z \rightarrow Z^{\text{ad}}$ . It has the following properties:

- (1.2)    i) if  $G$  is of adjoint type,  $\mu = 1$  (cf. [16, (1.17B)]),  
           ii) if  $G' \rightarrow G$  is a finite covering, then  $\mu(G)$  divides  $\mu(G')$ ,  
           iii) if  $G = SU(n, 1)$ , then  $\mu = n + 1$ .

Let  $\rho$  be an irreducible representation of  $G$  on the finite-dimensional complex vector space  $V$ . We will say that  $(\rho, V)$  is a *real* representation if there is a  $G$ -invariant  $\mathbf{R}$ -subspace  $V_{\mathbf{R}}$  of  $V$  with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that  $G$  acts on  $V$  by extension of scalars. Under the subgroup  $Z$ , the representation necessarily splits into one-dimensional  $Z$ -invariant summands, on each of which  $Z$  acts by a character. We pick an isomorphism

$$(1.3) \quad \phi: Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of  $Z$  is free cyclic, with elements

$$\chi_n: Z \rightarrow S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

$$(1.4) \quad V\langle n \rangle = \{v \in V : \rho(z)v = \chi_n(z)v \quad \text{if} \quad z \in Z\},$$

so that

$$(1.5) \quad V = \bigoplus_{n \in \mathbf{Z}} V\langle n \rangle.$$

Each  $V\langle n \rangle$  is invariant under  $K$ . If  $(\tau, W)$  is a representation of  $K$ , then we define  $W\langle n \rangle$  as in (1.4); if  $W$  is irreducible, then  $W = W\langle n \rangle$  for some  $n$ ,

i.e.,  $Z$  acts by a single character. We will assume to have chosen the isomorphism (1.3) so that  $Z$  acts on  $\mathfrak{p}^+$  by a "positive" character.

(1.6) *Example.* Assuming that  $G$  is almost simple, we take  $V_{\mathbf{R}} = \mathfrak{g}$ , and  $\rho = \text{Ad}$ , the adjoint representation of  $G$ . Then  $\mathfrak{p}^+ = V\langle\mu\rangle$ ,  $\mathfrak{k}_{\mathbf{C}} = V\langle 0\rangle$ , and  $\mathfrak{p}^- = V\langle -\mu\rangle$ .

For an irreducible (finite-dimensional) representation of  $G$ , we also use  $\rho$  to denote the induced action of  $\mathfrak{g}$  on  $V$ . Because of the above description (1.6) of  $\text{Ad}$ , it is easy to see that the following hold:

- (1.7) i)  $\rho(\mathfrak{p}^+) V\langle n\rangle \subset V\langle n+\mu\rangle$ ,  $\rho(\mathfrak{k}) V\langle n\rangle \subset V\langle n\rangle$ ,  
 $\rho(\mathfrak{p}^-) V\langle n\rangle \subset V\langle n-\mu\rangle$ .  
 ii)  $\{n: V\langle n\rangle \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, \dots, \lambda - m\mu\}$   
 for some integers  $\lambda \geq 0, m \geq \mu^{-1}\lambda$ .  
 iii) If  $V$  is real, then for all  $n$ ,  

$$V\langle -n\rangle = \overline{V\langle n\rangle} \quad (\text{complex conjugate})$$
  
 and thus  $m\mu = 2\lambda$ .

((1.7 i) includes, in particular, the standard fact that  $\mathfrak{p}^+$  and  $\mathfrak{p}^-$  are Abelian Lie subalgebras of  $\mathfrak{g}_{\mathbf{C}}$ .)

For the general case, write

$$(1.8) \quad G = (\prod_{j=1}^l G_j) \times H,$$

where each  $G_j$  is almost simple and of non-compact Hermitian type, and  $H$  is compact.<sup>1)</sup> Let

$$K = (\prod_{j=1}^l K_j) \times H;$$

$Z = \prod_{j=1}^l Z_j$ ; and  $Z^{\text{ad}} = \prod_{j=1}^l Z_j^{\text{ad}}$ , a product of circles. Let  $\Delta^{\text{ad}}$  be the diagonal of  $Z^{\text{ad}}$ , and  $\Delta$  the inverse image of  $\Delta^{\text{ad}}$  in  $Z$ . One may proceed as before, if we replace  $Z$  by  $\Delta$ . Alternatively, every irreducible representation  $(\rho, V)$  of  $G$  decomposes as a tensor product

$$(\otimes_{j=1}^l (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of  $\Delta$ , the decomposition (1.5) of  $V$  is the tensor product of the

<sup>1)</sup> We allow compact factors because of (2.7).

corresponding decompositions of each  $V_j$  into character spaces under  $Z_j$ , tensored with the "trivial" factor  $W$ .

On  $V$  there exists a positive-definite Hermitian form (the *admissible inner product*)  $T(v, w)$  (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$(1.9) \quad \begin{aligned} \text{i) } T(\rho(k)v, \rho(k)w) &= T(v, w) & \text{if } k \in K \\ \text{ii) } T(\rho(X)v, w) &= T(v, \rho(X)w) & \text{if } X \in \mathfrak{p}. \end{aligned}$$

This follows from the fact that  $\mathfrak{k} \oplus i\mathfrak{p}$  is a compact Lie algebra. If  $V$  is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on  $V_{\mathbf{R}}$ .

Let  $I$  denote the intersection of the kernels of all finite-dimensional representations of  $G$ . Then  $I$  is a central subgroup, and  $G/I$  admits the structure of a real (affine) algebraic group. Since we are interested in  $G$  only for its finite-dimensional representations and the symmetric space  $M$ , we may replace  $G$  by  $G/I$  and assume that  $G$  is an algebraic group. To get all of the representations of  $\mathfrak{g}$ , it is convenient in the abstract to replace  $G$  by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of  $G$ ); thus, we may and do assume that  $G$  is algebraically simply connected. We remark that by (1.2), the number  $\mu(G)$  can be arbitrarily large, even under this restriction.

Let, then,  $G_{\mathbf{C}}$  denote the set of complex points of  $G$ . It is a simply-connected complex Lie group with Lie algebra  $\mathfrak{g}_{\mathbf{C}}$ . Let  $K_{\mathbf{C}}$  denote the connected subgroup of  $G_{\mathbf{C}}$  with Lie algebra  $\mathfrak{k}_{\mathbf{C}}$ . By general theory (see [17, XVII.5]),  $K_{\mathbf{C}}$  is the universal complexification of  $K$ , and so, by definition, every representation of  $K$  extends to a holomorphic representation of  $K_{\mathbf{C}}$ .

Assume that  $M$  is Hermitian, and let  $P^+$  (resp.  $P^-$ ) denote the subgroup of  $G_{\mathbf{C}}$  corresponding to the subalgebra  $\mathfrak{p}^+$  (resp.  $\mathfrak{p}^-$ ) of  $\mathfrak{g}_{\mathbf{C}}$ . Then  $P^+K_{\mathbf{C}}P^-$  is an open subset of  $G_{\mathbf{C}}$  which contains  $G$  (see [4, p. 317]). Moreover,  $G \cap K_{\mathbf{C}}P^- = K$  (see [4, p. 318]), so the mapping of  $G \rightarrow G_{\mathbf{C}}$  induces a holomorphic embedding

$$(1.10) \quad M \rightarrow \check{M} = G_{\mathbf{C}}/Q;$$

as  $Q = K_{\mathbf{C}}P^-$  is a parabolic subgroup of  $G_{\mathbf{C}}$ ,  $\check{M}$  is compact and is known as the *compact dual* of  $M$ .