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## §1. PRELIMINARIES

Let G be a connected real semi-simple Lie group with finite center, K a maximal compact subgroup of G, and let  $g \supset \mathfrak{k}$  be the corresponding Lie algebras. For any sub-algebra  $\mathfrak{a} \subset \mathfrak{g}$ , we put

$$\mathfrak{a}_{\mathbf{C}} = \mathfrak{a} \otimes_{\mathbf{R}} \mathbf{C}$$
.

If B denotes the Killing form of g, B is negative-definite on  $\mathfrak{k}$ , and we let p denote the orthogonal complement under B of  $\mathfrak{k}$  in g. Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a so-called Cartan decomposition of g, and B is positive-definite on p.

Let M = G/K, the corresponding symmetric space. As  $[\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}$ , B defines an (Ad K)-invariant inner product on  $\mathfrak{p}$ ; and since we may identify  $\mathfrak{p}$  naturally as the tangent space to M at the identity coset  $x_0 = K$ , B determines a unique Riemannian metric on M which is invariant under the canonical left G-action.

Assume initially that M is an *irreducible* symmetric space. Then, if one wishes, G can be taken to be a non-compact almost simple group (i.e., g is a simple Lie algebra). In that case, the space M admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when f has a non-trivial center 3. In this case, dim 3 = 1, and  $Z = \exp 3$  is the identity component of the center of K. Let  $G^{ad}$  denote the adjoint group of G (i.e., the automorphism group of M) and let  $K^{ad}$ ,  $Z^{ad}$  be the corresponding subgroups of  $G^{ad}$ . A choice of  $z_0 \in Z^{ad}$  of order 4 (for which Ad  $(z_0^2)$  is a Cartan involution of g) determines an almost-complex structure on p:

 $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ 

with.

(1.1) 
$$\begin{cases} \mathfrak{p}^+ = \{ X \in \mathfrak{p}_{\mathbf{C}} \colon \mathrm{Ad} (z_0) X = iX \} \\ \mathfrak{p}^- = \{ X \in \mathfrak{p}_{\mathbf{C}} \colon \mathrm{Ad} (z_0) X = -iX \} \end{cases}$$

This determines, via left-translation under G, a Kählerian complex structure on M, such that the action of G is by holomorphic isometries.

For purposes of numeration, we define  $\mu = \mu(G)$  to be the degree of the covering map  $Z \to Z^{ad}$ . It has the following properties:

i) if G is of adjoint type, μ = 1 (cf. [16, (1.17B)]),
ii) if G' → G is a finite covering, then μ(G) divides μ(G'),
iii) if G = SU(n, 1), then μ = n + 1.

Let  $\rho$  be an irreducible representation of G on the finite-dimensional complex vector space V. We will say that  $(\rho, V)$  is a *real* representation if there is a Ginvariant **R**-subspace  $V_{\mathbf{R}}$  of V with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that G acts on V by extension of scalars. Under the subgroup Z, the representation necessarily splits into one-dimensional Z-invariant summands, on each of which Z acts by a character. We pick an isomorphism

(1.3) 
$$\phi: Z \simeq S^1 = \{ w \in \mathbf{C} : |w| = 1 \}.$$

The character group of Z is free cyclic, with elements

$$\chi_n: Z \to S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n$$

Let

(1.4) 
$$V < n > = \{ v \in V : \rho(z) \ v = \chi_n(z) \ v \quad \text{if} \quad z \in Z \},$$

so that

$$(1.5) V = \bigoplus_{n \in \mathbb{Z}} V < n > .$$

Each V < n > is invariant under K. If  $(\tau, W)$  is a representation of K, then we define W < n > as in (1.4); if W is irreducible, then W = W < n > for some n,

i.e., Z acts by a single character. We will assume to have chosen the isomorphism (1.3) so that Z acts on  $p^+$  by a "positive" character.

(1.6) *Example.* Assuming that G is almost simple, we take  $V_{\mathbf{R}} = g$ , and  $\rho = Ad$ , the adjoint representation of G. Then  $p^+ = V < \mu >$ ,  $\mathfrak{t}_{\mathbf{C}} = V < 0 >$ , and  $p^- = V < -\mu >$ .

For an irreducible (finite-dimensional) representation of G, we also use  $\rho$  to denote the induced action of g on V. Because of the above description (1.6) of Ad, it is easy to see that the following hold:

(1.7) i)  $\rho(\mathfrak{p}^+) V < n > \subset V < n + \mu >, \rho(\mathfrak{k}) V < n > \subset V < n >,$  $\rho(\mathfrak{p}^-) V < n > \subset V < n - \mu >.$ 

ii) 
$$\{n: V < n > \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, ..., \lambda - m\mu\}$$

for some integers  $\lambda \ge 0$ ,  $m \ge \mu^{-1}\lambda$ .

iii) If V is real, then for all n,

 $V < -n > = \overline{V < n >}$  (complex conjugate) and thus  $m\mu = 2\lambda$ .

((1.7 i) includes, in particular, the standard fact that  $p^+$  and  $p^-$  are Abelian Lie subalgebras of  $g_{c}$ .)

For the general case, write

$$(1.8) G = (\prod_{j=1}^{l} G_j) \times H$$

where each  $G_j$  is almost simple and of non-compact Hermitian type, and H is compact.<sup>1</sup>) Let

$$K = (\Pi_{j=1}^{l} K_{j}) \times H;$$

 $Z = \prod_{j=1}^{l} Z_{j}$ ; and  $Z^{ad} = \prod_{j=1}^{l} Z_{j}^{ad}$ , a product of circles. Let  $\Delta^{ad}$  be the diagonal of  $Z^{ad}$ , and  $\Delta$  the inverse image of  $\Delta^{ad}$  in Z. One may proceed as before, if we replace Z by  $\Delta$ . Alternatively, every irreducible representation ( $\rho$ , V) of G decomposes as a tensor product

$$\left(\bigotimes_{j=1}^{l}(\rho_{j}, V_{j})\right)\otimes(\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of  $\Delta$ , the decomposition (1.5) of V is the tensor product of the

<sup>&</sup>lt;sup>1</sup>) We allow compact factors because of (2.7).

corresponding decompositions of each  $V_j$  into character spaces under  $Z_j$ , tensored with the "trivial" factor  $W_j$ .

On V there exists a positive-definite Hermitian form (the *admissible inner* product) T (v, w) (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

(1.9) i) 
$$T(\rho(k) v, \rho(k) w) = T(v, w)$$
 if  $k \in K$   
ii)  $T(\rho(X) v, w) = T(v, \rho(X) w)$  if  $X \in \mathfrak{p}$ .

This follows from the fact that  $\mathfrak{t} \oplus i\mathfrak{p}$  is a compact Lie algebra. If V is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on  $V_{\mathbf{R}}$ .

Let I denote the intersection of the kernels of all finite-dimensional representations of G. Then I is a central subgroup, and G/I admits the structure of a real (affine) algebraic group. Since we are interested in G only for its finite-dimensional representations and the symmetric space M, we may replace G by G/I and assume that G is an algebraic group. To get all of the representations of g, it is convenient in the abstract to replace G by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of G); thus, we may and do assume that G is algebraically simply connected. We remark that by (1.2), the number  $\mu$  (G) can be arbitrarily large, even under this restriction.

Let, then,  $G_{\mathbf{c}}$  denote the set of complex points of G. It is a simply-connected complex Lie group with Lie algebra  $g_{\mathbf{c}}$ . Let  $K_{\mathbf{c}}$  denote the connected subgroup of  $G_{\mathbf{c}}$  with Lie algebra  $\mathfrak{t}_{\mathbf{c}}$ . By general theory (see [17, XVII.5]),  $K_{\mathbf{c}}$  is the universal complexification of K, and so, by definition, every representation of K extends to a holomorphic representation of  $K_{\mathbf{c}}$ .

Assume that *M* is Hermitian, and let  $P^+$  (resp.  $P^-$ ) denote the subgroup of  $G_{\mathbf{c}}$  corresponding to the subalgebra  $\mathfrak{p}^+$  (resp.  $\mathfrak{p}^-$ ) of  $\mathfrak{g}_{\mathbf{c}}$ . Then  $P^+K_{\mathbf{c}}P^-$  is an open subset of  $G_{\mathbf{c}}$  which contains *G* (see [4, p. 317]). Moreover,  $G \cap K_{\mathbf{c}} P^- = K$  (see [4, p. 318]), so the mapping of  $G \to G_{\mathbf{c}}$  induces a holomorphic embedding

(1.10)  $M \to \check{M} = G_{\mathbf{C}}/Q;$ 

as  $Q = K_{\mathbf{C}} P^{-}$  is a parabolic subgroup of  $G_{\mathbf{C}}$ ,  $\check{M}$  is compact and is known as the compact dual of M.

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