Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LOCALLY HOMOGENEOUS VARIATIONS OF HODGE STRUCTURE
Autor: Zucker, Steven
Kapitel: §1. Preliminaries
DOI: https://doi.org/10.5169/seals-51751

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§1. PRELIMINARIES

Let G be a connected real semi-simple Lie group with finite center, K a maximal compact subgroup of G, and let $\mathfrak{g} \supset \mathfrak{k}$ be the corresponding Lie algebras. For any sub-algebra $\mathfrak{a} \subset \mathfrak{g}$, we put

$$\mathfrak{a}_{\mathbf{C}} = \mathfrak{a} \otimes_{\mathbf{R}} \mathbf{C}.$$

If B denotes the Killing form of $\mathfrak{g}$, B is negative-definite on $\mathfrak{k}$, and we let $\mathfrak{p}$ denote the orthogonal complement under B of $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a so-called Cartan decomposition of $\mathfrak{g}$, and B is positive-definite on $\mathfrak{p}$.

Let $M = G/K$, the corresponding symmetric space. As $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, B defines an $(\mathrm{Ad}\ K)$-invariant inner product on $\mathfrak{p}$; and since we may identify $\mathfrak{p}$ naturally as the tangent space to M at the identity coset $x_0 = K$, B determines a unique Riemannian metric on M which is invariant under the canonical left G-action.

Assume initially that M is an *irreducible* symmetric space. Then, if one wishes, G can be taken to be a non-compact almost simple group (i.e., $\mathfrak{g}$ is a simple Lie algebra). In that case, the space M admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when $\mathfrak{k}$ has a non-trivial center $\mathfrak{z}$. In this case, $\dim \mathfrak{z} = 1$, and $Z = \exp \mathfrak{z}$ is the identity component of the center of K. Let G^{ad} denote the adjoint group of G (i.e., the automorphism group of M) and let K^{ad}, Z^{ad} be the corresponding subgroups of G^{ad}. A choice of $z_0 \in Z^{\mathrm{ad}}$ of order 4 (for which $\mathrm{Ad}\,(z_0^2)$ is a Cartan involution of $\mathfrak{g}$) determines an almost-complex structure on $\mathfrak{p}$:

$$\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$$

with

$$(1.1) \qquad \begin{cases} \mathfrak{p}^{+} = \{X \in \mathfrak{p}_{\mathbf{C}} : \mathrm{Ad}(z_0)\, X = iX\} \\ \mathfrak{p}^{-} = \{X \in \mathfrak{p}_{\mathbf{C}} : \mathrm{Ad}(z_0)\, X = -iX\} \end{cases}$$

This determines, via left-translation under G, a Kählerian complex structure on M, such that the action of G is by holomorphic isometries.

For purposes of numeration, we define $\mu = \mu(G)$ to be the degree of the covering map $Z \to Z^{ad}$. It has the following properties:

(1.2) i) if G is of adjoint type, $\mu = 1$ (cf. [16, (1.17B)]),

ii) if $G' \to G$ is a finite covering, then $\mu(G)$ divides $\mu(G')$,

iii) if $G = SU(n, 1)$, then $\mu = n + 1$.

Let ρ be an irreducible representation of G on the finite-dimensional complex vector space V. We will say that (ρ, V) is a *real* representation if there is a G-invariant $\mathbf{R}$-subspace $V_{\mathbf{R}}$ of V with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that G acts on V by extension of scalars. Under the subgroup Z, the representation necessarily splits into one-dimensional Z-invariant summands, on each of which Z acts by a character. We pick an isomorphism

$$(1.3) \qquad \phi : Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of Z is free cyclic, with elements

$$\chi_n : Z \to S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

$$(1.4) \qquad V\langle n\rangle = \{v \in V : \rho(z)\, v = \chi_n(z)\, v \quad \text{if} \quad z \in Z\},$$

so that

$$(1.5) \qquad V = \oplus_{n \in \mathbf{Z}} V\langle n\rangle.$$

Each $V\langle n\rangle$ is invariant under K. If (τ, W) is a representation of K, then we define $W\langle n\rangle$ as in (1.4); if W is irreducible, then $W = W\langle n\rangle$ for some n, i.e., Z acts by a single character. We will assume to have chosen the isomorphism (1.3) so that Z acts on $\mathfrak{p}^+$ by a "positive" character.

(1.6) *Example.* Assuming that G is almost simple, we take $V_{\mathbf{R}} = \mathfrak{g}$, and $\rho = \mathrm{Ad}$, the adjoint representation of G. Then $\mathfrak{p}^+ = V\langle\mu\rangle$, $\mathfrak{k}_{\mathbf{C}} = V\langle 0\rangle$, and $\mathfrak{p}^- = V\langle -\mu\rangle$.

For an irreducible (finite-dimensional) representation of G, we also use ρ to denote the induced action of $\mathfrak{g}$ on V. Because of the above description (1.6) of Ad, it is easy to see that the following hold:

(1.7) i) $\rho(\mathfrak{p}^+)\,V\langle n\rangle \subset V\langle n+\mu\rangle$, $\rho(\mathfrak{k})\,V\langle n\rangle \subset V\langle n\rangle$, $\rho(\mathfrak{p}^-)\,V\langle n\rangle \subset V\langle n-\mu\rangle$.

ii) $\{n\colon V\langle n\rangle \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, \ldots, \lambda - m\mu\}$ for some integers $\lambda \geqslant 0$, $m \geqslant \mu^{-1}\lambda$.

iii) If V is real, then for all n,

$$V\langle -n\rangle = \overline{V\langle n\rangle} \qquad \text{(complex conjugate)}$$

and thus $m\mu = 2\lambda$.

((1.7 i) includes, in particular, the standard fact that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Abelian Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$.)

For the general case, write

$$G = (\Pi_{j=1}^{l} G_j) \times H\,, \tag{1.8}$$

where each G_j is almost simple and of non-compact Hermitian type, and H is compact.[1]) Let

$$K = (\Pi_{j=1}^{l} K_j) \times H\,;$$

$Z = \Pi_{j=1}^{l} Z_j$; and $Z^{\mathrm{ad}} = \Pi_{j=1}^{l} Z_j^{\mathrm{ad}}$, a product of circles. Let Δ^{ad} be the diagonal of Z^{ad}, and Δ the inverse image of Δ^{ad} in Z. One may proceed as before, if we replace Z by Δ. Alternatively, every irreducible representation (ρ, V) of G decomposes as a tensor product

$$(\otimes_{j=1}^{l} (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of Δ, the decomposition (1.5) of V is the tensor product of the

[1]) We allow compact factors because of (2.7).

corresponding decompositions of each V_j into character spaces under Z_j, tensored with the "trivial" factor W.

On V there exists a positive-definite Hermitian form (the *admissible inner product*) $T(v, w)$ (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$\text{(1.9)} \quad \begin{aligned} &\text{i) } T(\rho(k)v, \rho(k)w) = T(v, w) && \text{if } k \in K \\ &\text{ii) } T(\rho(X)v, w) = T(v, \rho(X)w) && \text{if } X \in \mathfrak{p}. \end{aligned}$$

This follows from the fact that $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie algebra. If V is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on $V_{\mathbf{R}}$.

Let I denote the intersection of the kernels of all finite-dimensional representations of G. Then I is a central subgroup, and G/I admits the structure of a real (affine) algebraic group. Since we are interested in G only for its finite-dimensional representations and the symmetric space M, we may replace G by G/I and assume that G is an algebraic group. To get all of the representations of $\mathfrak{g}$, it is convenient in the abstract to replace G by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of G); thus, *we may and do assume that* G *is algebraically simply connected.* We remark that by (1.2), the number $\mu(G)$ can be arbitrarily large, even under this restriction.

Let, then, $G_{\mathbf{C}}$ denote the set of complex points of G. It is a simply-connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let $K_{\mathbf{C}}$ denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. By general theory (see [17, XVII.5]), $K_{\mathbf{C}}$ is the universal complexification of K, and so, by definition, every representation of K extends to a holomorphic representation of $K_{\mathbf{C}}$.

Assume that M is Hermitian, and let P^+ (resp. P^-) denote the subgroup of $G_{\mathbf{C}}$ corresponding to the subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) of $\mathfrak{g}_{\mathbf{C}}$. Then $P^+K_{\mathbf{C}}P^-$ is an open subset of $G_{\mathbf{C}}$ which contains G (see [4, p. 317]). Moreover, $G \cap K_{\mathbf{C}} P^- = K$ (see [4, p. 318]), so the mapping of $G \to G_{\mathbf{C}}$ induces a holomorphic embedding

$$\text{(1.10)} \qquad M \to \check{M} = G_{\mathbf{C}}/Q;$$

as $Q = K_{\mathbf{C}} P^-$ is a parabolic subgroup of $G_{\mathbf{C}}$, $\check{M}$ is compact and is known as the *compact dual* of M.