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EXAMPLE 4: IMPLICIT FUNCTION THEOREM. Consider an open subset $U \subset \mathbf{C}^2$ containing $(z_0, w_0) \in \mathbf{C}^2$ and a holomorphic function $f : U \to \mathbf{C}$. The classical existence and uniqueness theorem concerning the implicit function equation $f(z, w) = 0$ reads as follows. Assume that $f(z_0, w_0) = 0$ and $f'_w(z_0, w_0) \neq 0$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \to \mathbf{C}$ such that $g(z_0) = w_0$, $(z, g(z)) \in U$ and $f[z, g(z)] = 0$ if $z \in B_\delta(z_0)$. Moreover, if for some such δ, we have two holomorphic functions $g_j : B_\delta(z_0) \to \mathbf{C}$ such that $g_j(z_0) = w_o$, $(z, g_j(z)) \in U$ and $f[z, g_j(z)] = 0$ if $z \in B_\delta(z_0)$, where $j = 1, 2$, then $g_1(z) = g_2(z)$ for $z \in B_\delta(z_0)$. By keeping z_0, w_0 fixed, we would like to have a terminology allowing us to assert that the solution $w = g(z)$ passing through (z_0, w_0) of $f(z, w) = 0$ varies holomorphically with $f(z, w)$. This is done in elementary courses as follows. Consider an open subset $V \subset \mathbf{C}^3$ containing $(z_0, w_0, \lambda_0) \in \mathbf{C}^3$ and a holomorphic function $f : V \to \mathbf{C}$. The classical theorem concerning the implicit function equation $f(z, w, \lambda) = 0$ depending on the parameter λ reads as follows. Assume that $f(z_0, w_0, \lambda) = 0$ and $f'_w(z_0, w_0, \lambda) \neq 0$ if $(z_0, w_0, \lambda) \in V$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \times B_\delta(\lambda_0) \to \mathbf{C}$ such that $g(z_0, \lambda) = w_0$ if $\lambda \in B_\delta(\lambda_0)$, $(z, g(z, \lambda), \lambda) \in V$ and $f[z, g(z, \lambda), \lambda] = 0$ if $z \in B_\delta(z_0)$, $\lambda \in B_\delta(\lambda_0)$. Likewise if we have several parameters. We then say that, if an implicit function equation depends holomorphically on the variable, the unknown and the parameters, then its solution through a fixed point depends holomorphically on the variable and the parameters. See Example 4 of Section 3 below.

3. HOLOMORPHIC MAPPINGS

The topological vector spaces language is becoming a routine method of expression in Mathematics and certain of its applications, say to Mathematical Physics, Engineering and Economics. Our standard references are [6], [13], [15], [17], [31] and [32].

Let us recall that a complex topological vector space E is a vector space which at the same time is a topological space, such that the vector space operations $(x, y) \in E \times E \mapsto x + y \in E$ and $(\lambda, x) \in \mathbf{C} \times E \mapsto \lambda x \in E$ are continuous. A seminorm on a complex vector space E is a function $\alpha : E \mapsto \mathbf{R}_+$ such that $\alpha(x_1 + x_2) \leqslant \alpha(x_1) + \alpha(x_2)$ and $\alpha(\lambda x) = |\lambda| \cdot \alpha(x)$ $x_1, x_2, x \in E$, $\lambda \in \mathbf{C}$. We denote by $CS(E)$ the set of all continuous seminorms on a topological vector space E. If Γ is a nonvoid set of seminorms on a vector space E, we define the associated topology $\mathcal{T}_\Gamma$ on E by saying that $X \subset E$ is open if, whenever $x \in X$, there are $\alpha_1, \ldots, \alpha_n \in \Gamma$, $\varepsilon > 0$ for which $t \in E$, $\alpha_i(t-x) < \varepsilon$ for $i = 1, \ldots, n$ imply that $t \in X$. Then E is a topological vector space if endowed with $\mathscr{I}_\Gamma$. Which topological vector spaces do we get this way from arbitrary E and Γ? Well, $X \subset E$ is convex if, whenever $x_0, x_1 \in X$, $\lambda \in \mathbf{R}$, $0 \leqslant \lambda \leqslant 1$, then $(1-\lambda)x_0 + \lambda x_1 \in X$. A topological vector space E is locally convex if the convex neighborhoods of every $x \in E$ form a basis of neighborhoods of x; it suffices to check that at one point, say the origin. If Γ is a nonvoid set of seminorms on a vector space E, then E endowed with $\mathscr{I}_\Gamma$ is locally convex and $\Gamma \subset CS(E)$. Conversely, if E is a locally convex space, its topology $\mathscr{I}$ is associated to $\Gamma = CS(E)$, that is $\mathscr{I} = \mathscr{I}_\Gamma$. Hence locally convex spaces are just topological vector spaces whose topologies are defined by nonvoid sets of seminorms. There are basic results, such as the Hahn-Banach theorem, that are valid for locally convex spaces, but not necessarily for topological vector spaces. Fortunately, most topological vector spaces that we encounter are locally convex and have their topologies associated to sets Γ at sight. It is true that there are topological vector spaces that are not locally convex but are used, say in probability theory, typically $L^p(\mu)$ of a measure μ with $0 \leqslant p < 1$.

Fix the complex locally convex spaces E, F.

If $m = 1, 2, \ldots$, let $\mathscr{L}_a(^mE; F)$ be the vector space of all m-linear mappings of the cartesian power E^m to F; and $\mathscr{L}_{as}(^mE; F)$ be the vector subspace of all symmetric such mappings. Here (and in the sequel) the index "a" stands for "algebraic", continuity not being assumed. Let $\mathscr{L}(^mE; F)$ and $\mathscr{L}_s(^mE; F)$ be the vector subspaces of those $A \in \mathscr{L}_a(^mE; F)$ and $A \in \mathscr{L}_{as}(^mE; F)$ that are continuous, respectively. If $m = 0$, we set $\mathscr{L}_a(^0E; F) = \mathscr{L}_{as}(^0E; F) = \mathscr{L}(^0E; F) = \mathscr{L}_s(^0E; F) = F$.

Letting $A \in \mathscr{L}_a(^mE; F)$, $x \in E$, write $Ax^m = A(x, \ldots, x)$ if $m = 1, 2, \ldots$; and $Ax^0 = A$ if $m = 0$. To every such A, associate the mapping $\hat{A} : E \mapsto F$ defined by $\hat{A}(x) = Ax^m$ if $x \in E$. Call $\hat{A}$ the m-homogeneous polynomial associated to A. Denote by $\mathscr{P}_a(^mE; F)$ the vector space of all m-homogeneous polynomials of E to F associated to all $A \in \mathscr{L}_a(^mE; F)$; and by $\mathscr{P}(^mE; F)$ the vector subspace of all continuous such polynomials. The linear mappings $A \in \mathscr{L}_a(^mE; F) \mapsto \hat{A} \in \mathscr{P}_a(^mE; F)$ and $A \in \mathscr{L}(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$ are surjective. Moreover, the linear mappings $A \in \mathscr{L}_{as}(^mE; F) \mapsto \hat{A} \in \mathscr{P}_a(^mE; F)$ and $A \in \mathscr{L}_s(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$ are bijective.

Let $U \subset E$ be open and nonvoid. We say that $f : U \to F$ is holomorphic if, corresponding to every $\xi \in U$, there are Taylor coefficients $A_m \in \mathscr{L}_s(^mE; F)$

for $m = 0, 1, \ldots$ such that, for every $\beta \in CS(F)$, there is a neighborhood V of ξ in U for which

$$\lim_{m \to \infty} \beta \left[f(x) - \sum_{k=0}^{m} A_k (x-\xi)^k \right] = 0$$

uniformly for $x \in V$. Let $\mathscr{H}(U; F)$ be the vector space of all holomorphic mappings of U to F. If F is a normed space, the definition that f is holomorphic means that, corresponding to every $\xi \in U$, there are $A_m \in \mathscr{L}_s({}^mE; F)$ for $m = 0, 1, \ldots$ such that

$$f(x) = \sum_{m=0}^{\infty} A_m (x-\xi)^m ,$$

convergence being uniform for x in some neighborhood V of ξ in U. In general, the definition must be given as we phrased it.

If F is a Hausdorff space, the sequence (A_m) of Taylor coefficients of $f \in \mathscr{H}(U; F)$ at $\xi \in U$ is unique. Then

$$f(x) \cong \sum_{m=0}^{\infty} A_m (x-\xi)^m$$

is called the Taylor series of f at ξ, where $x \in U$. We define the m-differentials of f at ξ by

$$d^m f(\xi) = m! A_m , \quad \hat{d}^m f(\xi) = m! \hat{A}_m$$

considered as elements of $\mathscr{L}_s({}^mE; F)$ and $\rho({}^mE; F)$ respectively, for $m = 0, 1, \ldots$. The Taylor series of f at ξ becomes

$$\begin{aligned} f(x) &\cong \sum_{m=0}^{\infty} \frac{1}{m!} d^m f(\xi)(x-\xi)^m \\ &\cong \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(\xi)(x-\xi) . \end{aligned}$$

EXAMPLE 1: SPECTRAL THEORY. If F is a complex normed space and $f \in \mathscr{H}(\mathbf{C}; F)$ is bounded, the vector valued Liouville theorem asserts that f is a constant; this is proved exactly in the same way as when $F = \mathbf{C}$, that is, as in the classical Liouville theorem. This simple result was used in the proof of the Gelfand-Mazur theorem as given in Example 1 of Section 2. More generally, if E, F are complex locally convex spaces, F is a Hausdorff space, and $f \in \mathscr{H}(E; F)$ is bounded, that is $f(E)$ is bounded in F, then f is a constant; this is proved by a simple reduction to the case when $E = \mathbf{C}$ and F is a normed space. We recall that $Y \subset F$ is bounded in F if, for every neighborhood V of 0 in F, there is $\lambda \in \mathbf{C}$ such that $Y \subset \lambda V$.

EXAMPLE 2: OPERATIONAL CALCULUS. In the notation and terminology of Example 2 of Section 2, once $f \in \mathscr{H}(U; \mathbf{C})$ is fixed, the mapping $Z \in A^U \to f(Z) \in A$ is indeed holomorphic. All this becomes a more venturous enterprise in the more general case when, in the notation of Example 2 of Section 2, E is a locally convex space, or A is a locally convex algebra.

In order to reconsider Examples 3 and 4 of Section 2, we need to describe an important example of locally convex spaces, namely $\mathscr{H}(K; \mathbf{C})$, the space of germs of complex valued functions that are holomorphic around a fixed nonvoid compact subset K of $\mathbf{C}^n$. This example became a routine in Complex Analysis, Functional Analysis and applications. However, what happened historically may be described as follows. Fantappiè and others studied a lot the so-called analytic functionals, that is functions whose variable is an analytic (holomorphic) functions. Yet Fantappiè did not know how to introduce and use a natural topology on the spaces of holomorphic functions that he considered. Accordingly, he had to bypass this handicap to a certain extent. When Laurent Schwartz developed the theory of distributions, he naturally considered inductive (direct) limits. The most basic example of them in his theory is the following one. Once a nonvoid open subset $U \subset \mathbf{R}^n$ is fixed, the vector space $\mathscr{D}(U; \mathbf{C})$ of all infinitely differentiable complex valued functions on U with compact supports contained in U is to be looked upon as an inductive limit of the vector space $\mathscr{D}(U; \mathbf{C})$ of all such functions with supports contained in K, for any compact subset $K \subset U$. Next Dieudonné and Schwartz wrote an article on basic aspects of inductive limits of locally convex spaces. This led Dias, Grothendieck and Köthe simultaneously to define the natural topology on $\mathscr{H}(K; \mathbf{C})$ as follows.

Fix then a nonvoid compact subset $K \subset \mathbf{C}^n$ and consider the union

$$\mathscr{H}[K; \mathbf{C}] = \bigcup_{U \supset K} \mathscr{H}(U; \mathbf{C})$$

where U varies over all open subsets of $\mathbf{C}^n$ containing K. Define an equivalence relation modulo K on that union by considering $f_i : U_i \to \mathbf{C}$ $(i=1, 2)$ as equivalent if $U_i \subset \mathbf{C}^n$ is open containing K and $f_i \in \mathscr{H}(U_i; \mathbf{C})$, the set of points $x \in U_1 \cap U_2$ satisfying $f(x_1) = f(x_2)$ being a neighborhood of K in $\mathbf{C}^n$. Each equivalence class of $\mathscr{H}[K; \mathbf{C}]$ modulo such an equivalence relation is called a germ of holomorphic function around K. If $f \in \mathscr{H}[K; \mathbf{C}]$, we denote by $\tilde{f}_K$, or simply $\tilde{f}$, its equivalence class, that is, its germ modulo K. Call $\mathcal{H}(K; \mathbf{C})$ the quotient space of $\mathcal{H}[K; \mathbf{C}]$ modulo that equivalence relation. Then $\mathcal{H}(K; \mathbf{C})$ is a vector space in a unique way so that every mapping $f \in \mathcal{H}(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathcal{H}(K; \mathbf{C})$ is linear, where $U \subset \mathbf{C}^n$ is open containing K. Denote by $\mathcal{H}_B(U; \mathbf{C})$ the Banach space of all $f \in \mathcal{H}(U; \mathbf{C})$ that are bounded on U, where $\mathcal{H}_B(U; \mathbf{C})$ is endowed with the supremum norm. The natural topology on $\mathcal{H}(K; \mathbf{C})$ is defined by the following inductive limit procedure: it is the largest locally convex topology on $\mathcal{H}(K; \mathbf{C})$ such that each linear mapping $f \in \mathcal{H}_B(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathcal{H}(K; \mathbf{C})$ is continuous, for every open subset $U \subset \mathbf{C}^n$ containing K. We could also use an alternative form of this definition. The natural topology used on $\mathcal{H}(U; \mathbf{C})$ is the so-called compact-open topology. Then the same natural topology on $\mathcal{H}(K; \mathbf{C})$ may be defined by the following inductive limit procedure: it is the largest locally convex topology on $\mathcal{H}(K; \mathbf{C})$ such that each linear mapping $f \in \mathcal{H}(U; \mathbf{C}) \mapsto \tilde{f}_K \in \mathcal{H}(K; \mathbf{C})$ is continuous, for every open subset $U \subset \mathbf{C}^n$ containing K. If $K = \{z\}$ is reduced to a point $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, we write $\mathcal{H}(z; \mathbf{C}) = \mathcal{H}(z_1, \ldots, z_n; \mathbf{C})$ for $\mathcal{H}(\{z\}; \mathbf{C})$.

Example 3: Ordinary differential equations. Let us resume notation and terminology of Example 3 of Section 2. The classical existence and uniqueness theorem for ordinary differential equations allows us to associate to the germ $\tilde{f} \in \mathcal{H}(z_0, w_0; \mathbf{C})$ of $f \in \mathcal{H}(U; \mathbf{C})$ at (z_0, w_0) the germ $\tilde{g} \in \mathcal{H}(z_0; \mathbf{C})$ of $g \in \mathcal{H}(B_\delta(z_0); \mathbf{C})$ at z_0. It can be proved that the mapping $\tilde{f} \in \mathcal{H}(z_0, w_0; \mathbf{C}) \mapsto \tilde{g} \in \mathcal{H}(z_0; \mathbf{C})$ is holomorphic. It is really in this simple way that we should state that the solution passing through $(z_0; w_0)$ depends holomorphically on the differential equation. We see now how much exposition is needed to express that result in such a simple form. That is why we bypass such a language problem and state the result in the weaker classical form involving parameters; as a matter of fact, this is enough for certain purposes.

Example 4: Implicit function theorem. Let us resume notation and terminology of Example 4 of Section 2. Let $\mathcal{E}$ be the vector subspace of $\mathcal{H}(z_0, w_0; \mathbf{C})$ formed by all germs $\tilde{f} \in \mathcal{H}(z_0, w_0; \mathbf{C})$ of $f \in \mathcal{H}(U; \mathbf{C})$ at (z_0, w_0) satisfying $f(z_0, w_0) = 0$. Let $\mathcal{U}$ be the nonvoid open subset of $\mathscr{E}$ formed by those germs $\tilde{f}$ which, in addition to the above conditions, satisfy $f'_w(z_0, w_0) \neq 0$. The classical existence and uniqueness theorem for implicit function equations allows us to associate to the germ $\tilde{f} \in \mathscr{U}$ of $f \in \mathscr{H}(U; \mathbf{C})$ at (z_0, w_0) satisfying $f(z_0, w_0) = 0$, $f'_w(z_0, w_0) \neq 0$, the germ $\tilde{g} \in \mathscr{H}(z_0, \mathbf{C})$ of $g \in \mathscr{H}\left(B_\delta(z_0); \mathbf{C}\right)$ at z_0. It can be proved that the mapping $\tilde{f} \in \mathscr{U} \mapsto \tilde{g} \in \mathscr{H}(z_0; \mathbf{C})$ is holomorphic. We may repeat here some comments which are analogous to those made at the end of the above Example 3.

4. Concluding remarks

This article was written to attract prospective users in applications of holomorphy in infinite dimensions.

I have tried to illustrate through four very simple, classical examples, how the concept of holomorphic mappings in infinite dimensions comes up naturally in Analysis. The difference between Examples 1 and 2 on one side, and Examples 3 and 4 on the other side is striking: The first two examples seem very straightforward, while the last two examples look more sophisticated. However, sophistication in Mathematics is a matter of lack of habit; I personally am by now so used to dealing with germs of holomorphic functions that I no longer think of the last two examples as being sophisticated at all. Moreover, dealing long enough with any mathematical concept, particularly in applying it, leads to the development of a sort of intuition in that respect.

In 1963, I had my first opportunity of visiting Warsaw, and of talking leisurely to Mazur. I then played a little bit the role of a newspaper reporter and asked him if he, Banach and other members of the Polish group that developed Banach space theory, had specific applications in mind. Mazur answered, without any surprise to me as a mathematician, that the Polish group was guided by a conscience of the importance of Banach spaces in Mathematics proper. We witness nowadays how Banach spaces methods and results spread out in Mathematics and its applications. More accurately, Banach spaces have even been superseded by locally convex spaces for many of such goals. Psychologically, it is interesting to notice that the concept of a Banach space was also emphasized by Norbert Wiener; however Banach