2. Some classical motivations

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WHY HOLOMORPHY IN INFINITE DIMENSIONS?

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> Ein Mathematiker, der nicht etwas Poet ist, wird nie ein vollkommener Mathematiker
>
> KARL WEIERSTRASS

1. INTRODUCTION

The study of holomorphic functions in infinite dimensions is an objective as old in Mathematics as Functional Analysis, and as the idea of systems with an infinite number of degrees of freedom in Mechanics. It dates back to the end of the last century. The simple language of normed spaces and of topological vector spaces became a routine, as a suitable form of linear algebra in infinite dimensions to be used in Analysis, Geometry and applications. Thereafter, the theory of holomorphic mappings in infinite dimensions was properly developed as a confluence of ideas and methods originating mostly from several complex variables, manifold theory and Functional Analysis. Independently of that, users of sophisticated mathematical methods in applications have employed and furthered holomorphy in infinite dimensions, in fields such as Mathematical Physics and Electrical Engineering. The present expository article was written by aiming at the non-specialists, more exactly, at the non-mathematicians. We will use Weierstrass' definition as a model for the general case.

2. SOME CLASSICAL MOTIVATIONS

EXAMPLE 1: SPECTRAL THEORY. If $Z: E \to E$ is a linear operator on the complex vector space E of finite dimension $n = 1, 2, \ldots$, the homogeneous linear equation $Z(x) = \lambda x$ has at least some solution $x \in E$, $x \neq 0$ for at least some $\lambda \in \mathbf{C}$. Equivalently, there is at least some $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in the algebra $\mathscr{L}(E; E)$ of all linear operators on E, where I is the identity mapping of E; the set of all such λ has at most n elements. This fact is proved by noticing that $\lambda I - Z$ is not invertible if and only if, by taking determinants, the algebraic equation det $(\lambda I - Z) = 0$ is satisfied by λ. Now, we notice that det $(\lambda I - Z)$ is a polynomial in λ whose leading term is λ^n, hence of degree n. By the so-called fundamental theorem of Algebra, that algebraic equation has at least a solution $\lambda \in \mathbf{C}$; it is clear that it has at most n such solutions. This result is one of the starting points of Spectral Theory. A more general form of it is the following one. Let now $Z: E \to E$ be a continuous linear operator on the complex Banach space $E \neq 0$. There is at least some $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in the Banach algebra $\mathscr{L}(E; E)$ of all continuous linear operators on E, where I is the identity mapping of E. This result is no longer proved as in the finite dimensional situation, as we no longer have analogues of determinant theory and of the fundamental theorem of Algebra, as formerly. Since there is no difference in terms of difficulty in the exposition, we will explain this aspect in the more general language of Banach algebras. Let then A be a complex Banach algebra with unit $I \neq 0$; thus A is a Banach space and at the same time an algebra for which $\| XY \| \leqslant \| X \| \cdot \| Y \|$ if $X, Y \in A$, and $\| I \| = 1$. [For instance, if $E \neq 0$ is a complex Banach space, then $\mathscr{L}(E; E)$ is a complex Banach algebra with unit $I \neq 0$ in a natural way.] The spectrum spt (Z) of $Z \in A$ is the set of all $\lambda \in \mathbf{C}$ such that $\lambda I - Z$ is not invertible in A. The Gelfand-Mazur theorem states that spt (Z) is always nonvoid; it is clear that it is compact in $\mathbf{C}$. How can we prove such a result without analogues of determinant theory and of a fundamental theorem of Algebra? Surprisingly enough at first sight, this is accomplished through a seemingly isolated result in Complex Analysis, known as the Liouville theorem: if an entire complex valued function of a complex variable is bounded, then it must be a constant. As a matter of fact, it is immediately pointed out in Complex Analysis courses that a possible application of Liouville theorem is to a proof of the fundamental theorem of Algebra. Coming back to the Gelfand-Mazur theorem, its short but smart proof goes as follows. Assume that Z has a void spectrum. The vector valued function $\lambda \in \mathbf{C} \mapsto (\lambda I - Z)^{-1} \in A$ of a complex variable is entire, and it tends to zero at infinity. Thus the function in question must be a constant, by Liouville theorem, once it is entire and bounded; actually it must be identically zero as it is a constant and tends to zero at infinity. However, this is an absurdity as no inverse in A can be zero. The above proof calls for the need of a vector valued Liouville theorem of a complex variable, which not only is true but may be proved as easily as the scalar valued one. It is true that we may bypass the vector valued Liouville theorem by arguing as follows. For every continuous linear form φ on A, the scalar valued function $\lambda \in \mathbf{C} \mapsto \varphi\,[(\lambda I - Z)^{-1}] \in \mathbf{C}$ of a complex variable is entire, and it tends to zero at infinity. By the classical Liouville theorem, this function is identically zero for every such φ. By the Hahn-Banach theorem, if $X \in A$ satisfies $\varphi(X) = 0$ for every such φ, then $X = 0$. Thus $(\lambda I - Z)^{-1} = 0$ for all $\lambda \in \mathbf{C}$. However, this is an absurdity as no inverse in A can be zero. This equally nice proof of the Gelfand-Mazur theorem, via the classical Liouville theorem plus (the unnecessary use of) the Hahn-Banach theorem is like a good dessert whose recipe the cook does not tell us!... See Example 1 in Section 3 below.

EXAMPLE 2: OPERATIONAL CALCULUS. As in Example 1, we could consider the Banach algebra $\mathscr{L}(E; E)$ associated to a complex Banach space $E \neq 0$. Since there is no difference in terms of difficulty in the exposition, we will explain this aspect in the more general language of Banach algebras. Let then A be as in Example 1. If $f\colon \mathbf{C} \to \mathbf{C}$ is entire, we may consider its Taylor series

$$f(z) = \sum_{m=0}^{\infty} a_m (z - \xi)^m$$

about $\xi \in \mathbf{C}$, for any $z \in \mathbf{C}$, where $a_m = f^{(m)}(\xi)/m!$ for $m \in \mathbf{N}$. It is natural to define

$$f(Z) = \sum_{m=0}^{\infty} a_m (Z - \xi I)^m$$

for any $Z \in A$. It is easily checked that this definition makes sense, once $|a_m|^{1/m} \to 0$ as $m \to \infty$; and that $f(Z) \in A$ does not depend on the choice of ξ. Since the function $z \in \mathbf{C} \mapsto f(z) \in \mathbf{C}$ is entire, we would like to have a terminology allowing us to assert that the mapping $Z \in A \mapsto f(Z) \in A$ is entire too. For a change, consider now the nonvoid open subset $A^* \subset A$ formed by the invertible elements of A, and the nonvoid open subset $\mathbf{C}^* \subset \mathbf{C}$ formed by the nonzero elements of $\mathbf{C}$. Since the function $z \in \mathbf{C}^* \mapsto 1/z \in \mathbf{C}$ is holomorphic, we would like to have a terminology allowing us to assert that the mapping $Z \in A^* \mapsto Z^{-1} \in A$ is holomorphic too. More generally, let $\mathscr{H}(U; \mathbf{C})$ denote the algebra of all holomorphic functions $f\colon U \to \mathbf{C}$, where $U \subset \mathbf{C}$ is open nonvoid. If $f \in \mathscr{H}(U; \mathbf{C})$ and J is an oriented, rectifiable Jordan contour (formed by an exterior, counter-clockwise oriented, rectifiable Jordan curve and a finite number of interior, mutually exterior, clockwise oriented, rectifiable Jordan curves) fitted in U, we may consider the Cauchy integral

$$f(z) = \frac{1}{2\pi i} \int_{\lambda \in J} \frac{f(\lambda)}{\lambda - z} \, d\lambda$$

for any $z \in U$, provided z is surrounded by J. It is natural to define

$$f(Z) = \frac{1}{2\pi i} \int_{\lambda \in J} f(\lambda)(\lambda I - Z)^{-1} \, d\lambda$$

for any $Z \in A$ such that spt $(Z) \subset U$, provided spt (Z) is surrounded by J. It is easily checked that $f(Z) \in A$ does not depend on the choice of such J. The two previous cases are subsumed by the present one. Consider now the nonvoid open subset A^U of A formed by all $Z \in A$ such that spt $(Z) \subset U$. Since the function $z \in U \mapsto f(z) \in \mathbf{C}$ is holomorphic, we would like to have a terminology allowing us to assert that the mapping $Z \in A^U \mapsto f(Z) \in A$ is holomorphic too. This is indeed the case with the natural definition of holomorphic mappings between normed spaces. See Example 2 in Section 3 below.

EXAMPLE 3: ORDINARY DIFFERENTIAL EQUATIONS. Consider an open subset $U \subset \mathbf{C}^2$ containing $(z_0, w_0) \in \mathbf{C}^2$ and a holomorphic function $f : U \to \mathbf{C}$. The classical existence and uniqueness theorem concerning the ordinary differential equation $w' = f(z, w)$ reads as follows. If $\delta \in \mathbf{R}$, $\delta > 0$, let $B_\delta(z_0)$ be the set of all $z \in \mathbf{C}$ satisfying $|z - z_0| < \delta$. For some such δ, there is a holomorphic function $g : B_\delta(z_0) \to \mathbf{C}$ such that $g(z_0) = w_0$, $(z, g(z)) \in U$ and $g'(z) = f[z, g(z)]$ if $z \in B_\delta(z_0)$. Moreover, if for some such δ we have two holomorphic functions $g_j : B_\delta(z_0) \to \mathbf{C}$ such that $g_j(z_0) = w_0$, $(z, g_j(z)) \in U$ and $g'_j(z) = f[z, g_j(z)]$ if $z \in B_\delta(z_0)$, where $j = 1, 2$, then $g_1(z) = g_2(z)$ for $z \in B_\delta(z)$. By keeping z_0, w_0 fixed, we would like to have a terminology allowing us to assert that the solution $w = g(z)$ passing through (z_0, w_0) of $w' = f(z, w)$ varies holomorphically with $f(z, w)$. This is done in elementary courses as follows. Consider an open subset $V \subset \mathbf{C}^3$ containing $(z_0, w_0, \lambda_0) \in \mathbf{C}^3$ and a holomorphic function $f : V \to \mathbf{C}$. The classical theorem concerning the ordinary differential equation $w' = f(z, w, \lambda)$ depending on the parameter λ reads as follows. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \times B_\delta(\lambda_0) \to \mathbf{C}$ such that $g(z_0, \lambda) = w_0$ if $\lambda \in B_\delta(\lambda_0)$, $(z, g(z, \lambda), \lambda) \in V$ and $g'_z(z, \lambda) = f[z, g(z, \lambda), \lambda]$ if $z \in B_\delta(z_0)$, $\lambda \in B_\delta(\lambda_0)$. Likewise if we have several parameters. We then say that, if an ordinary differential equation depends holomorphically on the variable, the unknown and the parameters, then its solution through a fixed point depends holomorphically on the variable and the parameters. See Example 3 in Section 3 below.

EXAMPLE 4: IMPLICIT FUNCTION THEOREM. Consider an open subset $U \subset \mathbf{C}^2$ containing $(z_0, w_0) \in \mathbf{C}^2$ and a holomorphic function $f: U \to \mathbf{C}$. The classical existence and uniqueness theorem concerning the implicit function equation $f(z, w) = 0$ reads as follows. Assume that $f(z_0, w_0) = 0$ and $f'_w(z_0, w_0) \neq 0$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \to \mathbf{C}$ such that $g(z_0) = w_0$, $(z, g(z)) \in U$ and $f[z, g(z)] = 0$ if $z \in B_\delta(z_0)$. Moreover, if for some such δ, we have two holomorphic functions $g_j : B_\delta(z_0) \to \mathbf{C}$ such that $g_j(z_0) = w_o$, $(z, g_j(z)) \in U$ and $f[z, g_j(z)] = 0$ if $z \in B_\delta(z_0)$, where $j = 1, 2$, then $g_1(z) = g_2(z)$ for $z \in B_\delta(z_0)$. By keeping z_0, w_0 fixed, we would like to have a terminology allowing us to assert that the solution $w = g(z)$ passing through (z_0, w_0) of $f(z, w) = 0$ varies holomorphically with $f(z, w)$. This is done in elementary courses as follows. Consider an open subset $V \subset \mathbf{C}^3$ containing $(z_0, w_0, \lambda_0) \in \mathbf{C}^3$ and a holomorphic function $f: V \to \mathbf{C}$. The classical theorem concerning the implicit function equation $f(z, w, \lambda) = 0$ depending on the parameter λ reads as follows. Assume that $f(z_0, w_0, \lambda) = 0$ and $f'_w(z_0, w_0, \lambda) \neq 0$ if $(z_0, w_0, \lambda) \in V$. For some $\delta \in \mathbf{R}$, $\delta > 0$, there is a holomorphic function $g : B_\delta(z_0) \times B_\delta(\lambda_0) \to \mathbf{C}$ such that $g(z_0, \lambda) = w_0$ if $\lambda \in B_\delta(\lambda_0)$, $(z, g(z, \lambda), \lambda) \in V$ and $f[z, g(z, \lambda), \lambda] = 0$ if $z \in B_\delta(z_0)$, $\lambda \in B_\delta(\lambda_0)$. Likewise if we have several parameters. We then say that, if an implicit function equation depends holomorphically on the variable, the unknown and the parameters, then its solution through a fixed point depends holomorphically on the variable and the parameters. See Example 4 of Section 3 below.

3. HOLOMORPHIC MAPPINGS

The topological vector spaces language is becoming a routine method of expression in Mathematics and certain of its applications, say to Mathematical Physics, Engineering and Economics. Our standard references are [6], [13], [15], [17], [31] and [32].

Let us recall that a complex topological vector space E is a vector space which at the same time is a topological space, such that the vector space operations $(x, y) \in E \times E \mapsto x + y \in E$ and $(\lambda, x) \in \mathbf{C} \times E \mapsto \lambda x \in E$ are continuous. A seminorm on a complex vector space E is a function $\alpha : E \mapsto \mathbf{R}_+$ such that $\alpha(x_1 + x_2) \leqslant \alpha(x_1) + \alpha(x_2)$ and $\alpha(\lambda x) = |\lambda| \cdot \alpha(x)$ $x_1, x_2, x \in E$, $\lambda \in \mathbf{C}$. We denote by $CS(E)$ the set of all continuous seminorms on a topological vector space E. If Γ is a nonvoid set of seminorms on a vector space E, we define the associated topology $\mathcal{I}_\Gamma$ on E by saying