Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 26 (1980)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CHAPTER 14 OF RAMANUJAN'S SECOND NOTEBOOK

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DOI: https://doi.org/10.5169/seals-51058

CHAPTER 14 OF RAMANUJAN'S SECOND NOTEBOOK

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When Ramanujan died in 1920 he left behind three notebooks containing the statements of approximately 3000-4000 theorems. The second notebook is an enlarged edition of the first, and the third is short and fragmentary. The notebooks are thought to have been started in about 1903, and Ramanujan added to them until he left for England in early 1914. Many of the formulas that Ramanujan communicated to Hardy in his now famous letters [61] can be found in the notebooks. After Ramanujan's death, Hardy strongly urged the publication and editing of the notebooks. In 1929, G. N. Watson agreed to undertake this task with the assistance of B. M. Wilson. Possibly due to the premature death of Wilson, Watson never completed the monumental task that laid before him. However, Watson did write about 25 papers based on material found in the notebooks, mainly from Chapters 16-21 of the second notebook and from the heterogeneous material found at the end of the second notebook. The publication [62] of the notebooks was finally accomplished in 1957, but no editing whatsoever was undertaken. For a general description of the notebooks and their history, see a paper of Watson [74] or of the author [13].

Except for the papers of Watson and a paper of Hardy [28], [32] discussing a chapter on hypergeometric series, the contents of the notebooks have yet to be thoroughly examined. In the past several years, however, many authors have established certain formulas found in the notebooks. Most of these formulas are found in Chapter 14 of the second Notebook. In particular, the author [12] has shown that many of these formulas arise from a general transformation formula for a large class of analytic Eisenstein series. Nonetheless, the vast majority of the formulas in this chapter have not been previously proven in print. In this paper, we examine each of the 87 formulas found in Chapter 14. Our goal has been to prove each formula (when correct) or to give references to those formulas that have been previously established in print. A couple of the formulas are quite incorrect, and a few others need minor corrections. However, for the most part, the contents of Chapter 14 are quite accurate. Unfortunately, we have fallen short of our goal by one or two formulas. There is one formula (Entry 23i) which has resisted all attempts to prove it. The formula, in fact, is an approximate formula involving a certain "error term". Moreover, it is not clear at all how one should properly interpret this approximation. A companion formula (Entry 23ii) is equally vague in its meaning. However, by assuming that a certain inversion in order of summation is allowed, we can easily prove the formula, but with an "error term" that is identically zero. Ramanujan, clearly, had something more general in mind. The first corollary of Entry 1 was very difficult to prove. The author is extremely grateful to R. J. Evans for providing a proof of a corrected version of this formidable formula.

Chapter 14 is primarily concerned with identities involving infinite series. Hardy [61, p. xxv] remarked "There is always more in one of Ramanujan's formulae than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining." There could not be a more apt comment about Chapter 14 than this last sentence of Hardy. Some of the formulas were fairly easy to prove; others required considerable effort. As previously indicated, many of the formulas in Chapter 14 have their genesis in elliptic modular functions. A large number of formulas arise from partial fraction decompositions. Some formulae are instances of the Poisson summation formula. Six formulas lie in the realm of hypergeometric series. There are also a few integral evaluations.

It should be emphasized that Ramanujan very seldom stated any hypotheses for which his formulas were valid. Thus, the hypotheses accompanying the entries listed below are chiefly the author's. The formulas are not stated here in the style used by Ramanujan, but are given instead in contemporary notation. For example, Ramanujan rarely utilized a summation sign, and he employed the notation $\underline{|x}$ for $\Gamma(x+1)$. We also adhere to the contemporary even suffixed notation for the Bernoulli and Euler numbers, as found in [1, p. 804], for example. Also, those formulas which needed corrections are stated in corrected form.

The material in Chapter 14 is not organized particularly well. In presenting and discussing the contents of the chapter, it could be argued that a reorganization of the chapter's formulas is preferable. However, we have decided to discuss each formula in the order of its appearance. Not only is historical faithfulness preserved, but those who wish to consult the Notebooks while reading this paper should find their task made easier.

Very few claims are made that our proofs are like those found by Ramanujan. In fact, we make heavy use of Cauchy's residue theorem. As Hardy [31, p. 19] pointed out, Ramanujan never showed any interest in the theory of functions of a complex variable.

In the sequel, $R(f, z_0) = R(z_0)$ denotes the residue of f at a pole z_0. Also, $\chi(n)$ always denote the primitive character of modulus 4, i.e.,

$$\chi(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{0.1}$$

ENTRY 1. For $z^2 \neq -n(n+1)/2$, where n is a nonnegative integer, we have

$$z^{-2} \prod_{n=1}^{\infty} \left(1 + \frac{2z^2}{n(n+1)}\right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{z^2 + n(n+1)/2}. \tag{1.1}$$

Proof. From the partial fraction decomposition [75, p. 136]

$$\operatorname{sech} x = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 \pi^2 + 4x^2}, \tag{1.2}$$

we obtain, after some simplification,

$$2\pi \operatorname{sech}\left(\pi \sqrt{2z^2 - 1/4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{z^2 + n(n+1)/2}.$$

From the product expansion [23, p. 37]

$$\cosh z = \prod_{n=0}^{\infty} \left(1 + \frac{4z^2}{(2n+1)^2 \pi^2}\right)$$

and Wallis's product [23, p. 12]

$$\frac{\pi}{4} = \prod_{n=1}^{\infty} \frac{4n(n+1)}{(2n+1)^2},$$

we find that

$$\begin{aligned} 2\pi \operatorname{sech}\left(\pi \sqrt{2z^2 - 1/4}\right) &= \frac{\pi}{4} z^{-2} \prod_{n=1}^{\infty} \left(1 + \frac{8z^2 - 1}{(2n+1)^2}\right)^{-1} \\ &= z^{-2} \prod_{n=1}^{\infty} \left\{ \frac{(2n+1)^2}{4n(n+1)} \left(1 + \frac{8z^2 - 1}{(2n+1)^2}\right) \right\}^{-1} \\ &= z^{-2} \prod_{n=1}^{\infty} \left(1 + \frac{2z^2}{n(n+1)}\right)^{-1}. \end{aligned}$$

The result now follows.

Corollary of Entry 1. For $\operatorname{Re} z > 0$,

$$(1.3)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ (2n+1)}{\sqrt{n(n+1)}\,(e^{2\pi z\sqrt{n(n+1)}}-1)}$$

$$+\frac{1}{z}\sum_{n=1}^{\infty} \operatorname{sech}\left(\frac{\pi}{z}\sqrt{n^2-z^2/4}\right) = \frac{1}{2\pi z}+\frac{\pi z}{6}-C,$$

where

$$(1.4)\qquad C=\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}{}^{*}\ \frac{(-1)^{n+1}\,(2n+1)}{\sqrt{n(n+1)}},$$

where the * on the summation sign above indicates that the terms must be added in successive pairs in order for the series to converge.

We first show that the series defining C converges. We have

$$\frac{2n+1}{\sqrt{n(n+1)}}-\frac{2n+3}{\sqrt{(n+1)(n+2)}}$$

$$=\frac{1}{\sqrt{n+1}}\left\{\frac{2n+1}{\sqrt{n}}-\frac{2n+3}{\sqrt{n}}\left(1-\frac{1}{n}+\frac{3}{2n^2}+O\left(\frac{1}{n^3}\right)\right)\right\}$$

$$=\frac{1}{\sqrt{n+1}}\left\{O\left(\frac{1}{n^{5/2}}\right)\right\}=O(n^{-3}),$$

and so C is well defined.

Formula (1.3) does not agree with the corresponding entry in the Notebooks in that Ramanujan claims that C should be replaced by

$$(1.5)\qquad \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1+2\sqrt{n(n+1)}}$$

$$=1-\frac{\pi}{8}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\{2n+1+2\sqrt{n(n+1)}\}^2}.$$

It is not difficult to prove the foregoing equality. Indeed, let C' denote the left side of (1.5). Using Gregory's series for $\pi/4$, we find that

$$C'=1-\frac{\pi}{8}+\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{2(2n+1)}+\frac{(-1)^{n+1}}{2n+1+2\sqrt{n(n+1)}}\right\}$$

$$= 1 - \frac{\pi}{8} + \sum_{n=1}^{\infty} (-1)^n \frac{\{2\sqrt{n(n+1)} - (2n+1)\}}{2(2n+1)\{2n + 1 + 2\sqrt{n(n+1)}\}}$$

$$= 1 - \frac{\pi}{8} + \sum_{n=1}^{\infty} (-1)^n \frac{\{4n(n+1) - (2n+1)^2\}}{2(2n+1)\{2n + 1 + 2\sqrt{n(n+1)}\}^2},$$

and (1.5) easily follows.

Calculations of J. Hill first demonstrated that the constant given by Ramanujan is incorrect. In fact, $C' = .61144169\ldots$, while $C = .54661949\ldots$. The formula for C given in (1.4) can be transformed into another formula which exhibits Ramanujan's error. Letting $a_n = (2n+1)/\sqrt{n(n+1)}$, we have

$$C = \frac{1}{2} + \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (a_n - a_{n+1})$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \{(a_n - 1) - (a_{n+1} - 1)\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \left\{\frac{n + 1/2}{\sqrt{n(n+1)}} - 1\right\}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n + 1 - 2\sqrt{n(n+1)}}{2\sqrt{n(n+1)}}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2\sqrt{n(n+1)}\,(2n+1+2\sqrt{n(n+1)})}.$$

Comparing the above with (1.5), we find that Ramanujan neglected a factor of $2\sqrt{n(n+1)}$ in the denominators of the summands on the left side of (1.5).

After stating the Corollary of Entry 1, Ramanujan declares "Similarly any function whose denominator is in the form of a product can be expressed as the sum of partial fractions and many other theorems may be deduced from the result." But nonetheless, we have been unable to prove that (1.3) is a corollary of (1.1). The following proof of (1.3) is due to R. J. Evans.

Proof of Corollary to Entry 1. We prove the result for $z = x > 0$; the more general result will then hold by analytic continuation.

For $n \geqslant 1$, let a_n be as defined above and put

$$f_n(x) = \frac{1}{e^{2\pi x\sqrt{n(n+1)}} - 1} - \frac{1}{2\pi x\sqrt{n(n+1)}}.$$

Thus,

$$(1.6)\qquad \sum_{n=1}^{\infty} (-1)^{n+1} a_n f_n(x)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{\sqrt{n(n+1)}\,(e^{2\pi x\sqrt{n(n+1)}}-1)}$$

$$- \frac{1}{2\pi x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+1)}{n(n+1)} .$$

By combining successive terms we find after an elementary calculation that

$$(1.7)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+1)}{n(n+1)} = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 .$$

Putting (1.7) into (1.6) and comparing the resulting equality with (1.3), we find that we must show that

$$(1.8)\quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n f_n(x) + \frac{1}{x} \sum_{n=1}^{\infty} \operatorname{sech}\left(\frac{\pi}{x}\sqrt{n^2 - x^2/4}\right) - \frac{\pi x}{6} = -C ,$$

where

$$C = \frac{1}{2} + \frac{1}{2} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} (a_n - a_{n+1}) .$$

From [75, p. 136],

$$(1.9)\qquad \frac{1}{e^x - 1} = \frac{1}{2}\coth\left(\frac{x}{2}\right) - \frac{1}{2} = \frac{1}{x} - \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2x}{x^2 + 4\pi^2 m^2} .$$

Using (1.2) and (1.9) in (1.8) and then simplifying, we find that

$$(1.10)\qquad \sum_{n=1}^{\infty} \left\{ \frac{1}{2}(-1)^n a_n + \frac{x}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{n+1}(2n+1)}{n(n+1)x^2 + m^2} \right\}$$

$$= -C + \frac{\pi x}{6} - \frac{x}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{m(m+1)x^2 + n^2}$$

$$= -C + \frac{x}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(2m+1)}{m(m+1)x^2 + n^2} .$$

Letting

$$B(m,n) = \frac{2n+1}{n(n+1)x^2 + m^2} ,$$

we see that (1.10) may be written as

$$(1.11) \qquad \frac{x}{\pi} \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \sum_{m=1}^{\infty} \{B(m, n-1) - B(m, n)\}$$

$$= -\frac{1}{2} + \frac{x}{\pi} \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \{B(m, n-1) - B(m, n)\}.$$

A brief calculation gives

$$B(m, n-1) - B(m, n) = \frac{2n^2x^2 - 2m^2}{(m^2 + n^2x^2 - nx^2)(m^2 + n^2x^2 + nx^2)}.$$

Replacing x by $x/2$, we see then that (1.11) is equivalent to

$$(1.12) \qquad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2 - n^2x^4/4}$$

$$= -\frac{\pi}{2x} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2 - n^2x^4/4}.$$

By a brief calculation,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2 - n^2x^4/4} - \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2} \right\}$$

is seen to be an absolutely convergent double series, and so an inversion in order of summation is justified. Thus, (1.12) is seen to be equivalent to

$$(1.13) \qquad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2} = -\frac{\pi}{2x} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2x^2 - m^2}{(m^2 + n^2x^2)^2}$$

$$= -\frac{\pi}{2x} - x^{-2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^2x^{-2} - m^2}{(m^2 + n^2x^{-2})^2},$$

where on the right side we have replaced the indices m and n by n and m, respectively. Let the left side of (1.13) be denoted by $F(x)$. Thus, (1.13) may be rewritten as

$$(1.14) \qquad F(x) + x^{-2}F(1/x) = -\pi/(2x).$$

Now return to (1.9). Replace x by $2\pi nx$ and differentiate the extremal sides with respect to x. After some simplification, we find that

$$\text{(1.15)} \qquad \frac{2\pi^2}{(e^{\pi nx}-e^{-\pi nx})^2} - \frac{1}{2n^2x^2}$$

$$= \sum_{m=1}^{\infty} \frac{2n^2x^2}{(n^2x^2+m^2)^2} - \sum_{m=1}^{\infty} \frac{1}{n^2x^2+m^2} = \sum_{m=1}^{\infty} \frac{n^2x^2-m^2}{(n^2x^2+m^2)^2}.$$

Summing both sides of (1.15) on n, $1 \leqslant n < \infty$, we deduce that

$$\frac{1}{2}\pi^2 \sum_{n=1}^{\infty} \operatorname{csch}^2(\pi nx) - \frac{\pi^2}{12x^2} = F(x).$$

Thus, (1.14) is seen to be equivalent to

$$\pi x \sum_{n=1}^{\infty} \operatorname{csch}^2(\pi nx) + \frac{\pi}{x} \sum_{n=1}^{\infty} \operatorname{csch}^2(\pi n/x)$$

$$= -1 + \frac{\pi}{6}\left(x + \frac{1}{x}\right).$$

If we put $\alpha = \pi x$ and $\beta = \pi/x$, we find that for α, $\beta > 0$ and $\alpha\beta = \pi^2$,

$$\text{(1.16)} \qquad \alpha \sum_{n=1}^{\infty} \operatorname{csch}^2(\alpha n) + \beta \sum_{n=1}^{\infty} \operatorname{csch}^2(\beta n) = -1 + (\alpha+\beta)/6.$$

In summary, we have shown that (1.3) is equivalent to (1.16). But the author [12, Proposition 2.25] has previously proved (1.16), and hence the proof is complete.

Observe that (1.13) provides a beautiful example of a nonabsolutely convergent double series whose order of summation cannot be inverted.

ENTRY 2. Let m, n, x, and y be complex numbers. Suppose that $\Gamma(1+xz)$ and $\Gamma(1+yz)$ have no coincident poles and that $z = 1$ is not a pole of either. Then if $\operatorname{Re}(m+n) > 0$,

$$\text{(2.1)} \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,\Gamma(1-ky/x)}{\Gamma(m-k+1)\,\Gamma(n+1-ky/x)\,\Gamma(k)\,(x+k)}$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,\Gamma(1-kx/y)}{\Gamma(n-k+1)\,\Gamma(m+1-kx/y)\,\Gamma(k)\,(y+k)}$$

$$= \frac{\Gamma(x+1)\,\Gamma(y+1)}{\Gamma(x+m+1)\,\Gamma(y+n+1)}.$$

Proof. Let

$$f(z) = \frac{\Gamma(1+xz)\,\Gamma(1+yz)}{\Gamma(m+xz+1)\,\Gamma(n+yz+1)\,(z-1)}.$$

Then f has poles at $z = 1$, $-j/x$, and $-k/y$, where $1 \leqslant j, k < \infty$, and all poles are simple by hypothesis. Routine calculations yield

$$R(1) = \frac{\Gamma(x+1)\,\Gamma(y+1)}{\Gamma(m+x+1)\,\Gamma(n+y+1)},$$

$$R(-j/x) = \frac{(-1)^j\,\Gamma(1-jy/x)}{\Gamma(m-j+1)\,\Gamma(n-jy/x+1)(j+x)\,\Gamma(j)},$$

and

$$R(-k/y) = \frac{(-1)^k\,\Gamma(1-kx/y)}{\Gamma(m-kx/y+1)\,\Gamma(n-k+1)(k+y)\,\Gamma(k)}.$$

Let C_N be a positively oriented square centered at the origin and with vertical and horizontal sides of length $2N$. We shall let N tend to ∞ on some countable subset of the positive real numbers chosen so that the sides of C_N never get closer than some fixed positive distance from the set of poles of f. Using Stirling's formula, we find that

$$f(z) = O(|z|^{-\mathrm{Re}(m+n)-1}),$$

as $|z|$ tends to ∞. Hence, if $\mathrm{Re}\,(m+n) > 0$, we deduce that

$$\int_{C_N} f(z)\,dz = o(1), \tag{2.2}$$

as N tends to ∞.

Now integrate f over C_N and apply the residue theorem. Let N tend to ∞ and use (2.2). We then deduce (2.1) immediately.

COROLLARY 1 OF ENTRY 2. Let m, n, and x be complex numbers such that x is not an integer and that $\mathrm{Re}\,(m+n) > -1$. Then

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(x+k)\,\Gamma(m+1-k)\,\Gamma(n+1+k)} = \frac{\pi}{\sin(\pi x)\,\Gamma(m+x+1)\,\Gamma(n-x+1)}. \tag{2.3}$$

COROLLARY 2 OF ENTRY 2. Let α and β be complex numbers with $\mathrm{Re}\,(\alpha+\beta) > 0$. Then

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\,\Gamma(\alpha-k)\,\Gamma(\beta+k+1)} \tag{2.4}$$

$$+ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\,\Gamma(\beta-k)\,\Gamma(\alpha+k+1)} = \frac{\pi}{2\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)} .$$

Corollaries 1 and 2 are not really corollaries of Entry 2. Ramanujan evidently means to imply that the *proofs* of the present results are very much like the proof of the preceding theorem.

Proof of Corollary 1. Let

$$f(z) = \frac{\pi}{\sin(\pi z)(z+x)\,\Gamma(m+1-z)\,\Gamma(n+1+z)} .$$

Observe that f has a simple pole at $z = -x$ and at each integer k. Routine calculations give

$$R(-x) = -\frac{\pi}{\sin(\pi x)\,\Gamma(m+1+x)\,\Gamma(n+1-x)}$$

and

$$R(k) = \frac{(-1)^k}{(k+x)\,\Gamma(m+1-k)\,\Gamma(n+1+k)} .$$

Let C_N be the positively oriented square centered at the origin with vertical and horizontal sides passing through $\pm(N+1/2)$ and $\pm(N+1/2)\,i$, respectively, where N is a positive integer. By Stirling's formula,

$$f(z) = O(|z|^{-\mathrm{Re}(m+n)-2}),$$

as $|z|$ tends to ∞. Hence, for $\mathrm{Re}(m+n) > -1$,

$$\int_{C_N} f(z)\,dz = o(1), \tag{2.5}$$

as N tends to ∞. Apply the residue theorem to the integral of f over C_N. Let N tend to ∞. Using (2.5), we deduce (2.3) at once.

Proof of Corollary 2. Integrate

$$\frac{\pi}{\sin(\pi z)(z-1/2)\,\Gamma(\alpha+z)\,\Gamma(\beta-z+1)}$$

over the same square as in the foregoing proof. The present proof follows along precisely the same lines, and we omit it.

A second proof can be given as follows. Let the left side of (2.4) be denoted by $g(\alpha, \beta)$. After a little manipulation, we see that $g(\alpha, \beta)$ may be written as

$$g(\alpha, \beta) = \frac{\sin(\pi\alpha)}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2}+k\right) \Gamma(1-\alpha+k)}{\Gamma\left(\frac{3}{2}+k\right) \Gamma(1+\beta+k)}, \tag{2.6}$$

which converges absolutely for $\mathrm{Re}(\alpha+\beta) > 0$ by Stirling's formula. Now apply Dougall's formula [33, p. 52] to the right side of (2.6) to obtain

$$g(\alpha, \beta) = \frac{\pi}{2\Gamma\left(\alpha+\frac{1}{2}\right) \Gamma\left(\beta+\frac{1}{2}\right)}.$$

As pointed out by Hardy [28], [32, pp. 505-516], Ramanujan independently discovered Dougall's formula and made several applications of it. Quite possibly then, Ramanujan established Corollary 2 by the argument employed in the second proof above. Corollary 1 may also be proved with the aid of Dougall's theorem. However, Dougall's theorem is not applicable to Entry 2. The next entry is also an instance of Dougall's theorem.

ENTRY 3. Let α, β, γ, and δ be complex numbers such that $\mathrm{Re}(\alpha+\beta+\gamma+\delta) > -1$. Then

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha-k+1)\, \Gamma(\beta-k+1)\, \Gamma(\gamma+k+1)\, \Gamma(\delta+k+1)} \\ &+ \sum_{k=1}^{\infty} \frac{1}{\Gamma(\alpha+k+1)\, \Gamma(\beta+k+1)\, \Gamma(\gamma-k+1)\, \Gamma(\delta-k+1)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma+\delta+1)}{\Gamma(\alpha+\gamma+1)\, \Gamma(\beta+\gamma+1)\, \Gamma(\alpha+\delta+1)\, \Gamma(\beta+\delta+1)}. \end{aligned} \tag{3.1}$$

Proof. The left side of (3.1) may be written as

$$\frac{\sin(\pi\alpha)\, \sin(\pi\beta)}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{\Gamma(k-\alpha)\, \Gamma(k-\beta)}{\Gamma(\gamma+k+1)\, \Gamma(\delta+k+1)},$$

which converges absolutely for $\mathrm{Re}(\alpha+\beta+\gamma+\delta) > -1$ by Stirling's formula. A straightforward application of Dougall's formula [33, p. 52] yields (3.1) immediately.

ENTRY 4. If $z \neq me^{\pm \pi i/3}$, where m is a nonzero integer, then

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2 + z^4/n^2} = \frac{\pi}{2z\sqrt{3}} \frac{\sinh(\pi z\sqrt{3}) - \sqrt{3}\sin(\pi z)}{\cosh(\pi z\sqrt{3}) - \cos(\pi z)}. \tag{4.1}$$

Proof. Let $f(z)$ denote the right side of (4.1). We expand f into partial fractions. Since

$$\cosh(\pi z\sqrt{3}) - \cos(\pi z) = 2\sin(\pi z e^{\pi i/3})\sin(\pi z e^{-\pi i/3}),$$

f has simple poles at $z = ne^{\pm \pi i/3}$ for each nonzero integer n. Now

$$R(ne^{-\pi i/3}) = \frac{\sinh(\pi n e^{-\pi i/3}\sqrt{3}) - \sqrt{3}\sin(\pi n e^{-\pi i/3})}{4n(-1)^n\sqrt{3}\sin(\pi n e^{-2\pi i/3})}.$$

Note that $R(-ne^{-\pi i/3}) = -R(ne^{-\pi i/3})$. The residues of the poles at $\pm ne^{\pi i/3}$ are obtained by replacing $e^{-\pi i/3}$ by $e^{\pi i/3}$ above. For each positive integer n, the sum of the principal parts for the four poles $\pm ne^{\pm \pi i/3}$ is then

$$\frac{(-1)^n}{2\sqrt{3}} \left\{ \frac{e^{-\pi i/3}\{\sinh(\pi n e^{-\pi i/3}\sqrt{3}) - \sqrt{3}\sin(\pi n e^{-\pi i/3})\}}{\sin(\pi n e^{-2\pi i/3})(z^2 - n^2 e^{-2\pi i/3})} + \frac{e^{\pi i/3}\{\sinh(\pi n e^{\pi i/3}\sqrt{3}) - \sqrt{3}\sin(\pi n e^{\pi i/3})\}}{\sin(\pi n e^{2\pi i/3})(z^2 - n^2 e^{2\pi i/3})} \right\}. \tag{4.2}$$

Elementary calculations give

$$\sin(\pi n e^{\pm 2\pi i/3}) = \begin{cases} \pm i(-1)^m \sinh(\pi m\sqrt{3}), & \text{if} \quad n = 2m, \\ (-1)^{m+1}\cosh(\pi n\sqrt{3}/2), & \text{if} \quad n = 2m+1, \end{cases}$$

and

$$e^{\pm \pi i/3}\{\sinh(\pi n e^{\pm \pi i/3}\sqrt{3}) - \sqrt{3}\sin(\pi n e^{\pm \pi i/3})\}$$
$$= \begin{cases} 2(-1)^m \sinh(\pi m\sqrt{3}), & \text{if} \quad n = 2m, \\ \pm 2i(-1)^{m+1}\cosh(\pi n\sqrt{3}/2), & \text{if} \quad n = 2m+1. \end{cases}$$

Using the above calculations, we find that (4.2) simplifies to $n^2/(z^4 + n^2z^2 + n^4)$ for both n even and n odd. Hence,

$$f(z) = \sum_{n=1}^{\infty} \frac{n^2}{z^4 + n^2z^2 + n^4} + g(z),$$

where g is entire. However, as $|z| \to \infty$, we clearly see that $g(z) \to 0$.

Thus, g is a bounded entire function. By Liouville's theorem, $g(z)$ is constant, and this constant is obviously 0. Hence, the proof is complete.

COROLLARY TO ENTRY 4. For each nonzero integer n,

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + (2n)^2 + (2n)^4/k^2} = \frac{1}{12n^2} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2 + 3n^2}.$$

Proof. In the derivation below, we shall employ (1.9) and [75, p. 136]

$$\operatorname{csch}(\pi z) = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 + k^2}.$$

In Entry 4 let $z = 2n$ to get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2 + (2n)^2 + (2n)^4/k^2} &= \frac{\pi}{4n\sqrt{3}} \{\coth(2\pi n\sqrt{3}) + \operatorname{csch}(2\pi n\sqrt{3})\} \\ &= \frac{1}{12n^2} + \sum_{k=1}^{\infty} \frac{1}{12n^2 + k^2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{12n^2 + k^2}, \end{aligned}$$

and the result follows.

ENTRY 5i. Let $0 < x < \pi/(n+1/2)$, where n is a positive integer. Then

$$\sum_{k=1}^{\infty} \frac{\sin^{2n+1}(kx)}{k} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}.$$

Proof. Since [23, p. 25]

$$\sin^{2n+1} x = 2^{-2n} \sum_{j=0}^{n} (-1)^{n+j} \binom{2n+1}{j} \sin\{(2n+1-2j)x\},$$

we have

$$\sum_{k=1}^{\infty} \frac{\sin^{2n+1}(kx)}{k} \tag{5.1}$$
$$= \frac{(-1)^n}{2^{2n}} \sum_{j=0}^{n} (-1)^j \binom{2n+1}{j} \sum_{k=1}^{\infty} \frac{\sin\{(2n+1-2j)kx\}}{k}.$$

Using the familiar result [23, p. 38]

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2}, \qquad 0 < x < 2\pi, \tag{5.2}$$

we find that, for $0 < x < \pi/(n+1/2)$,

$$(5.3) \quad \sum_{k=1}^{\infty} \frac{\sin^{2n+1}(kx)}{k} = \frac{(-1)^n}{2^{2n}} \sum_{j=0}^{n} (-1)^j \binom{2n+1}{j} \frac{\pi - (2n+1-2j)x}{2}$$

$$= \frac{\pi}{2^{2n+1}} \binom{2n}{n} - \frac{(-1)^n}{2^{2n+1}} \sum_{j=0}^{n} (-1)^j \binom{2n+1}{j} (2n+1-2j)x,$$

where we used the evaluation [23, p. 3]

$$(5.4) \quad \sum_{j=0}^{m} (-1)^j \binom{k}{j} = (-1)^m \binom{k-1}{m},$$

with $m = n$ and $k = 2n+1$.

We next show that the sum on the far right side of (5.3) vanishes. We have

$$2 \sum_{j=0}^{n} (-1)^j \binom{2n+1}{j} (2n+1-2j) = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} (2n+1-2j)$$

$$= (2n+1) \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} - 2 \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} j = 0,$$

where we have used (5.4) and [23, p. 4]

$$\sum_{j=0}^{n} (-1)^j j \binom{n}{j} = 0.$$

Hence, from (5.3),

$$\sum_{k=1}^{\infty} \frac{\sin^{2n+1}(kx)}{k} = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

which is easily seen to be equivalent to the desired result by the Legendre duplication formula.

ENTRY 5ii. Let $0 \leqslant x \leqslant \pi/(n+1)$, where n is a positive integer. Then

$$(5.5) \quad \sum_{k=1}^{\infty} \frac{\sin^{2n+2}(kx)}{k^2} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} x.$$

Proof. Let $f(x)$ denote the left side of (5.5). Since [23, p. 25]

$$\sin^{2n+2} x = 2^{-2n-2} \left\{ \sum_{j=0}^{n} (-1)^{n+1+j} 2 \binom{2n+2}{j} \cos \{2(n+1-j)x\} + \binom{2n+2}{n+1} \right\},$$

we find that

$$(5.6) \quad f(x) = \frac{(-1)^{n+1}}{2^{2n+1}} \sum_{j=0}^{n} (-1)^j \binom{2n+2}{j} \sum_{k=1}^{\infty} \frac{\cos\{2(n+1-j)kx\}}{k^2} + \frac{1}{2^{2n+2}} \binom{2n+2}{n+1} \frac{\pi^2}{6}.$$

Now [23, p. 39]

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}, \qquad 0 \leqslant x \leqslant 2\pi.$$

Employing the above and (5.4) with $m = n$ and $k = 2n + 2$, we find that (5.6) becomes

$$(5.7) \quad f(x) = \frac{(-1)^{n+1}}{2^{2n+1}} \sum_{j=0}^{n} (-1)^j \binom{2n+2}{j} \{(n+1-j)^2 x^2 - \pi(n+1-j)x\} - \frac{1}{2^{2n+1}} \binom{2n+1}{n} \frac{\pi^2}{6} + \frac{1}{2^{2n+2}} \binom{2n+2}{n+1} \frac{\pi^2}{6},$$

where $0 \leqslant x \leqslant \pi/(n+1)$. First,

$$(5.8) \quad 2 \sum_{j=0}^{n} (-1)^j \binom{2n+2}{j} (n+1-j)^2 = \sum_{j=0}^{2n+2} (-1)^j \binom{2n+2}{j} (n+1-j)^2 = \sum_{j=0}^{2n+2} (-1)^j \binom{2n+2}{j} j^2 = 0.$$

Next, from two applications of (5.4), we find that

$$(5.9) \quad \sum_{j=0}^{n} (-1)^j \binom{2n+2}{j} (n+1-j) = (2n+2) \sum_{j=0}^{n} (-1)^j \binom{2n+1}{j} - (n+1) \sum_{j=0}^{n} (-1)^j \binom{2n+2}{j}$$
$$= (2n+2)(-1)^n \binom{2n}{n} - (n+1)(-1)^n \binom{2n+1}{n} = (-1)^n \binom{2n}{n}.$$

Substituting (5.8) and (5.9) into (5.7), we find that

$$f(x) = \frac{\pi x}{2^{2n+1}} \binom{2n}{n},$$

which is again equivalent to the desired result by the Legendre duplication formula.

ENTRY 6. For $n > 0$, let

$$\phi(\beta) = \prod_{k=0}^{\infty} \left\{ 1 + \left(\frac{\beta}{n+k} \right)^2 \right\}^{-1}.$$

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$. Then

$$\sqrt{\alpha} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sech}^{2n}(\alpha k) \right\}$$

$$= \frac{\Gamma(n)}{\Gamma(n+1/2)} \sqrt{\beta} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \phi(\beta k) \right\}.$$

Proof. Recall the Poisson summation formula. If f is a continuous function of bounded variation on $[a, b]$, then

$$\sum_{a \leq k \leq b}{}' f(k) = \int_a^b f(x)\, dx + 2 \sum_{k=1}^{\infty} \int_a^b f(x) \cos(2\pi kx)\, dx, \tag{6.1}$$

where the prime $'$ on the summation sign at the left indicates that if a or b is an integer, then only $\frac{1}{2} f(a)$ or $\frac{1}{2} f(b)$, respectively, is counted.

Now $\phi(x)$ was studied by Ramanujan in [58], [61, pp. 53-58]. On page 54 of [61] Ramanujan remarks that

$$\phi(x) = \frac{\Gamma(n+ix)\,\Gamma(n-ix)}{\Gamma^2(n)}.$$

This is not too difficult to prove; use the Weierstrass product formula for the quotient of Γ-functions above, and after considerable simplification, the desired equality follows. We shall apply (6.1) with $f(x) = \phi(\beta x)$, $a = 0$, and $b = \infty$. By using Stirling's formula for $|\Gamma(n+ix)\,\Gamma(n-ix)|$, as x tends to ∞, we easily justify letting b tend to ∞. Furthermore, for $m \geqslant 0$ and $n > 0$ [58], [61, p. 53],

$$\int_0^{\infty} \phi(x) \cos(2mx)\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1/2)}{\Gamma(n)} \operatorname{sech}^{2n} m.$$

Hence, since $\phi(0) = 1$, (6.1) yields

$$\frac{1}{2} + \sum_{k=1}^{\infty} \phi(\beta k) = \frac{1}{\beta}\int_0^{\infty} \phi(x)\, dx + \frac{2}{\beta}\sum_{k=1}^{\infty} \int_0^{\infty} \phi(x) \cos(2\pi kx/\beta)\, dx$$

$$= \sqrt{\frac{\alpha}{\beta}}\, \frac{\Gamma(n+1/2)}{\Gamma(n)} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sech}^{2n}(\pi k/\beta) \right\},$$

which is easily seen to be equivalent to the desired result.

ENTRY 7. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$ and let z be an arbitrary complex number. Then

$$e^{z^2/4}\sqrt{\alpha}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\alpha^2 k^2} \cos(\alpha z k)\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\beta^2 k^2} \cosh(\beta z k)\right\}.$$

Proof. Apply the Poisson formula (6.1) with $f(x) = \exp(-\alpha^2 x^2) \cos(\alpha z x)$, $a = 0$, and $b = \infty$. Now [23, p. 480],

$$\int_0^{\infty} e^{-cx^2} \cos(rx)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{c}}\, e^{-r^2/(4c)}, \tag{7.1}$$

where $\operatorname{Re} c > 0$ and r is arbitrary. With the use of the above evaluation, all of the calculations are quite routine, and the desired formula follows with no difficulty.

COROLLARY TO ENTRY 7. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$. Then

$$\sqrt{\alpha}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\alpha^2 k^2}\right\} = \sqrt{\beta}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\beta^2 k^2}\right\}.$$

Proof. Let $z = 0$ in Entry 7.

Note that the above is simply the functional equation for the classical theta-function.

ENTRY 8i. Let $\alpha, \beta, n > 0$ with $\alpha\beta = \pi$ and $0 < \beta n < \pi$. Then

$$\alpha \sum_{k=1}^{\infty} \frac{\sinh(2\alpha n k)}{e^{2\alpha^2 k} - 1} + \beta \sum_{k=1}^{\infty} \frac{\sin(2\beta n k)}{e^{2\beta^2 k} - 1}$$

$$= \frac{1}{4}\alpha \coth(\alpha n) - \frac{1}{4}\beta \cot(\beta n) - \frac{1}{2} n.$$

Entry 8i arises from the transformation formulae of a function akin to the logarithm of the Dedekind eta-function. The first proof of Entry 8i preceded that by Ramanujan and was found by Schlömilch [69], [70, p. 156]. Later proofs have been given by Rao and Ayyar [63], Lagrange [44], and the author [12, equation (3.31)], [8, equation (11.21)].

ENTRY 8ii. Let $\alpha, \beta, n > 0$ with $\alpha\beta = \pi$ and $0 < \alpha n < \pi$. Then

$$2 \sum_{k=1}^{\infty} \frac{\cos(2\alpha n k)}{k(e^{2\alpha^2 k} - 1)} - 2 \sum_{k=1}^{\infty} \frac{\cosh(2\beta n k)}{k(e^{2\beta^2 k} - 1)}$$

$$= n^2 - \frac{1}{6}(\alpha^2 - \beta^2) + \operatorname{Log}\left\{\frac{\sin(\alpha n)}{\sinh(\beta n)}\right\}.$$

Entry 8ii arises from the transformation formulae of a function which generalizes the logarithm of the Dedekind eta-function. Proofs have been given by Lagrange [44] and the author [12, Proposition 3.4].

ENTRY 8iii. Let $\alpha, \beta, n, r, t > 0$ with $\alpha\beta = \pi$, $r = n\beta$, and $t = \pi/\beta^2$. Let C be the positively oriented parallelogram with vertices $\pm i$ and $\pm t$. Let $\phi(z)$ be entire. Let m be a positive integer and put $M = m + 1/2$. Define

$$f_m(z) = \frac{\phi(rMz)}{z(e^{-2\pi Mz} - 1)(e^{2\pi i Mz/t} - 1)},$$

and assume that $f_m(z)$ tends to 0 boundedly on $C' = C - \{\pm i, \pm t\}$ as m tends to ∞. Then

$$(8.1) \quad \sum_{k=1}^{\infty} \frac{\phi(\alpha n k) + \phi(-\alpha n k)}{k(e^{2\alpha^2 k} - 1)} + \sum_{k=1}^{\infty} \frac{\phi(\alpha n k)}{k}$$

$$- \sum_{k=1}^{\infty} \frac{\phi(\beta n k i) + \phi(-\beta n k i)}{k(e^{2\beta^2 k} - 1)} - \sum_{k=1}^{\infty} \frac{\phi(\beta n k i)}{k}$$

$$= \frac{\pi i \phi(0)}{2} - \frac{\alpha^2 \phi(0)}{6} + \frac{\beta^2 \phi(0)}{6} - \frac{\alpha n \phi'(0)}{2} + \frac{\beta n i \phi'(0)}{2} - \frac{n^2 \phi''(0)}{4},$$

provided that all series above converge.

The obviously very restrictive hypotheses on ϕ are of a technical nature. We could state these hypotheses more specifically, but an even lengthier statement of the theorem would be necessary.

Proof. We integrate $f_m(z)$ over C. On the interior of C, f_m has simple poles at $z = \pm ik/M$ and at $z = \pm kt/M$, $1 \leqslant k \leqslant m$. Also, there is a triple pole at $z = 0$. Straightforward calculations give

$$R(ik/M) = \frac{\phi(rki)}{2\pi ik}\left\{\frac{1}{e^{2\pi k/t} - 1} + 1\right\},$$

$$R(-ik/M) = \frac{\phi(-rki)}{2\pi ik(e^{2\pi k/t} - 1)},$$

$$R(kt/M) = -\frac{\phi(rkt)}{2\pi ik}\left\{\frac{1}{e^{2\pi kt} - 1} + 1\right\},$$

and

$$R(-kt/M) = -\frac{\phi(-rkt)}{2\pi ik(e^{2\pi kt} - 1)},$$

where $1 \leqslant k \leqslant m$. Now,

$$f_m(z) = \frac{it}{(2\pi M)^2 z^3}\left\{\phi(0) + \phi'(0)\, rMz + \frac{1}{2}\phi''(0)(rMz)^2 + \dots\right\}$$

$$\cdot\left\{1 + \pi Mz + \frac{1}{3}(\pi Mz)^2 + \dots\right\}\left\{1 - \frac{\pi i Mz}{t} - \frac{1}{3}\left(\frac{\pi Mz}{t}\right)^2 + \dots\right\},$$

and so

$$R(0) = \frac{\phi(0)}{4} + \frac{it\phi(0)}{12} - \frac{i\phi(0)}{12t} + \frac{irt\phi'(0)}{4\pi} + \frac{r\phi'(0)}{4\pi} + \frac{ir^2t\phi''(0)}{8\pi^2}.$$

Applying the residue theorem and letting M tend to ∞, we find that

$$(8.2)\qquad \lim_{m\to\infty}\int_C f_m(z)\,dz = \sum_{k=1}^{\infty}\frac{\phi(rki) + \phi(-rki)}{k(e^{2\pi k/t} - 1)} + \sum_{k=1}^{\infty}\frac{\phi(rki)}{k}$$

$$- \sum_{k=1}^{\infty}\frac{\phi(rkt) + \phi(-rkt)}{k(e^{2\pi kt} - 1)} - \sum_{k=1}^{\infty}\frac{\phi(rkt)}{k} + \frac{\pi i\phi(0)}{2}$$

$$- \frac{\pi t\phi(0)}{6} + \frac{\pi\phi(0)}{6t} - \frac{rt\phi'(0)}{2} + \frac{ir\phi'(0)}{2} - \frac{r^2t\phi''(0)}{4\pi}.$$

By our hypotheses and the bounded convergence theorem, the limit on the left side of (8.2) is 0. Substituting $r = n\beta$ and $t = \pi/\beta^2$ in (8.2) and rearranging, we deduce (8.1).

We next show that Entry 8ii is a special instance of Entry 8iii.

Let $\phi(z) = \exp(2iz)$. Thus, $\phi(\alpha nk) + \phi(-\alpha nk) = 2\cos(2\alpha nk)$ and $\phi(\beta nki) + \phi(-\beta nki) = 2\cosh(2\beta nk)$. Since $0 < \alpha n < \pi$, by [23, p. 38] and (5.2), we have

$$\sum_{k=1}^{\infty} \frac{\phi(\alpha nk)}{k} = \sum_{k=1}^{\infty} \frac{e^{2\alpha nki}}{k} = -\operatorname{Log}\{2\sin(\alpha n)\} + i\,\frac{\pi - 2\alpha n}{2}.$$

Secondly, an elementary calculation gives

$$\sum_{k=1}^{\infty} \frac{\phi(\beta nki)}{k} = \sum_{k=1}^{\infty} \frac{e^{-2\beta nk}}{k} = \beta n - \operatorname{Log}\{2\sinh(\beta n)\}.$$

Thus,

$$\sum_{k=1}^{\infty} \frac{\phi(\beta nki)}{k} - \sum_{k=1}^{\infty} \frac{\phi(\alpha nk)}{k} + \frac{\pi i\phi(0)}{2} - \frac{\alpha^2\phi(0)}{6} + \frac{\beta^2\phi(0)}{6}$$
$$- \frac{\alpha n\phi'(0)}{2} + \frac{\beta ni\phi'(0)}{2} - \frac{n^2\phi''(0)}{4}$$
$$= \operatorname{Log}\left\{\frac{\sin(\alpha n)}{\sinh(\beta n)}\right\} + n^2 - \frac{1}{6}\alpha^2 + \frac{1}{6}\beta^2.$$

Hence, formally, Entry 8ii follows readily from Entry 8iii.

It remains to check the hypotheses concerning the parallelogram C. This is easily done by parameterizing each side of C. In the first quadrant, $f_m(z)$ trivially tends to 0 boundedly on C'. The same is true on C' in the second quadrant, but the hypothesis $r > 0$ is needed. Since $0 < \alpha n < \pi$, $f_m(z)$ tends to 0 boundedly on that part of C' in the lower half-plane.

Corollary i of Entry 8iii. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then

$$\alpha \sum_{k=1}^{\infty} \frac{k}{e^{2\alpha k} - 1} + \beta \sum_{k=1}^{\infty} \frac{k}{e^{2\beta k} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \tag{8.3}$$

This entry is really not a corollary of Entry 8iii; however, a proof can be given along somewhat the same lines.

Formula (8.3) was first established by Schlömilch [69], [70, p. 157]. Other proofs have been given by Malurkar [49], Rao and Ayyar [63], Lagrange [44], Grosswald [25], and the author [12, Proposition 2.11], [8, equation (11.7)]. In essence, (8.3) was also established by Hurwitz [34], [35] and Guinand [26], although neither author explicitly states the formula.

COROLLARY ii OF COROLLARY i. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then

$$e^{(\alpha-\beta)/12} = \left(\frac{\alpha}{\beta}\right)^{1/4} \prod_{k=1}^{\infty} \frac{1 - e^{-2\alpha k}}{1 - e^{-2\beta k}}.$$

Proof. Let $u, v > 0$ with $uv = \pi^2$. Write Corollary i in the form

$$\sum_{k=1}^{\infty} \frac{ke^{-2uk}}{1 - e^{-2uk}} + \frac{v}{u} \sum_{k=1}^{\infty} \frac{ke^{-2vk}}{1 - e^{-2vk}} = \frac{1}{24} + \frac{v/u}{24} - \frac{1}{4u}.$$

Integrate both sides of the above with respect to u over the interval $[\pi, \alpha]$ to obtain

$$\frac{1}{2} \sum_{k=1}^{\infty} \operatorname{Log} \frac{1 - e^{-2\alpha k}}{1 - e^{-2\pi k}} + \sum_{k=1}^{\infty} k \int_{\pi}^{\alpha} \frac{e^{-2vk}(v/u)}{1 - e^{-2vk}} \, du$$

$$= \frac{\alpha - \pi}{24} + \frac{1}{24} \int_{\pi}^{\alpha} (v/u) \, du + \frac{1}{4} \operatorname{Log}(\pi/\alpha).$$

In the integrals that remain, make the change of variable $u = \pi^2/v$. By the hypothesis, the limits π and α are transformed into π and β, respectively. Thus, the above becomes

$$\frac{1}{2} \sum_{k=1}^{\infty} \operatorname{Log} \frac{1 - e^{-2\alpha k}}{1 - e^{-2\beta k}} = \frac{\alpha - \beta}{24} + \frac{1}{8} \operatorname{Log}(\beta/\alpha).$$

Multiplying both sides by 2 and then exponentiating both sides yields the desired result.

EXAMPLE. We have

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k} - 1} = \frac{1}{24} - \frac{1}{8\pi}. \tag{8.4}$$

This example is obtained from Corollary i by setting $\alpha = \beta = \pi$. Ramanujan stated (8.4) as a problem in [56], [61, p. 326]. He later gave a proof of (8.4) in [57, p. 361], [61, p. 34] by using some formulae from the theory of elliptic functions. But, as already indicated, (8.4) was first established by Schlömilch [69], [70, p. 157]. Proofs of (8.4) have also been given by Krishnamaghari [43], Watson [73], Sandham [66], Lewittes [46], [47], and Ling [48], in addition to the authors listed after Corollary i.

ENTRY 9i. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi/2$. Let $h > 0$ be chosen so that $h/\alpha > 1$ and h/α is not an odd integer. Let m be the greatest odd integer that is less than h/α. Let n be an arbitrary real number. Let $\phi(x)$ be continuous and of bounded variation on $[0, h]$ and define

$$\psi(t) = \int_0^h \phi(x) \cos(tx)\, dx . \tag{9.1}$$

If χ is defined by (0.1), then

$$\alpha \sum_{k=1}^{m} \chi(k) \sin(\alpha nk)\, \phi(\alpha k) = \frac{1}{2} \sum_{k=1}^{\infty} \chi(k) \{ \psi(\beta k - n) - \psi(\beta k + n) \} .$$

Proof. Let f be a continuous function of bounded variation on $[a, b]$. Then the Poisson formula for sine transforms [72, p. 66]

$$\sum_{a \leq k \leq b}{}' \chi(k) f(k) = \sum_{k=1}^{\infty} \int_a^b f(x) \sin(\pi k x/2)\, dx \tag{9.2}$$

is valid, where the prime $'$ on the summation sign on the left side has the same meaning as in (6.1). Let $f(x) = \sin(\alpha nx)\, \phi(\alpha x)$, $a = 0$, and $b = h/\alpha$. Then

$$\begin{aligned} &\sum_{k=1}^{m} \chi(k) \sin(\alpha nk)\, \phi(\alpha k) \\ &= \sum_{k=1}^{\infty} \chi(k) \int_0^{h/\alpha} \sin(\alpha nx)\, \phi(\alpha x) \sin(\pi kx/2)\, dx . \end{aligned} \tag{9.3}$$

The integrals on the right side of (9.3) are easily calculated by (9.1) to complete the proof.

ENTRY 9ii. Let a, β, h, m, n, and ϕ satisfy the same hypotheses as in Entry 9i. Define

$$\psi(t) = \int_0^h \phi(x) \sin(tx)\, dx .$$

Then

$$\alpha \sum_{k=1}^{m} \chi(k) \cos(\alpha nk)\, \phi(\alpha k) = \frac{1}{2} \sum_{k=1}^{\infty} \chi(k) \{ \psi(\beta k - n) + \psi(\beta k + n) \} .$$

Proof. The proof is completely analogous to that for Entry 9i.

Ramanujan stated Entries 9i and 9ii with the extra condition $|n| < \beta$, but this hypothesis does not seem necessary.

ENTRY 10. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi/4$, and let z be an arbitrary complex number. Then

$$e^{z^2/4}\sqrt{\alpha}\sum_{k=1}^{\infty}\chi(k)\,e^{-\alpha^2k^2}\sin(\alpha zk)$$
$$=\sqrt{\beta}\sum_{k=1}^{\infty}\chi(k)\,e^{-\beta^2k^2}\sinh(\beta zk)\,.$$

Entry 10 should be compared with Entry 7.

Proof. Apply (9.2) with $f(x) = e^{-\alpha^2x^2}\sin(\alpha zx)$, $a = 0$, and $b = \infty$. By (7.1),

$$\int_0^{\infty} e^{-\alpha^2x^2}\sin(\alpha zx)\sin(\pi kx/2)\,dx$$
$$=\frac{1}{4\alpha}\sqrt{\pi}\left(e^{-(\alpha z-\pi k/2)^2/(4\alpha^2)}-e^{-(\alpha z+\pi k/2)^2/(4\alpha^2)}\right)$$
$$=\sqrt{\beta/\alpha}\,e^{-k^2\beta^2-z^2/4}\sinh(\beta zk)\,.$$

The entry now readily follows.

ENTRY 11. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$, and let n be real with $|n| < \beta/2$. Then

$$\alpha\left\{\frac{1}{4}\sec(\alpha n)+\sum_{k=1}^{\infty}\chi(k)\frac{\cos(\alpha nk)}{e^{\alpha^2k}-1}\right\} \tag{11.1}$$
$$=\beta\left\{\frac{1}{4}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cosh(2\beta nk)}{\cosh(\beta^2k)}\right\}.$$

Proof. We shall use a transformation formula, Theorem 3 (i), from the author's paper [10]. We refer the reader to [10] for all notation used below.

Let $V(z) = -1/z$, $r_1 = 0$, and $-1 < r_2 = r < 0$. Then $R_1 = r$, $R_2 = 0$, and $\rho = 0$. Also, let $s = -N = 1$. By [10, equation (4.5)], we find that

$$f^*(z, 1; 0, r; 1, \mu) = 2\pi i\left\{\frac{1}{8z}+B_1\left(\frac{\mu-r}{4}\right)\right\},$$

where $B_1(x)$ denotes the first Bernoulli polynomial. It follows that

$$\sum_{\mu=0}^{3} \chi(\mu) f^*(z, 1; 0, r; 1, \mu) = -\pi i . \tag{11.2}$$

We next calculate, for $\operatorname{Im} z > 0$,

$$\begin{aligned} &H_2(z, 1; \chi; 0, r) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \chi(k) e^{\pi i k(mz+r)/2} + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \chi(k) e^{\pi i k(mz-r)/2} \\ &= 2 \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\pi k r/2)}{e^{-\pi i k z/2} - 1} . \end{aligned}$$

Hence,

$$\begin{aligned} &z^{-1}(-2\pi i/4) G(\chi) H_2(-1/z, 1; \chi; 0, r) \\ &= \frac{2\pi}{z} \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\pi k r/2)}{e^{\pi i k/(2z)} - 1} . \end{aligned} \tag{11.3}$$

Next, we calculate, for $\operatorname{Im} z > 0$,

$$\begin{aligned} &-2\pi i H_1(z, 1; \chi; r, 0) \\ &= -2\pi i \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \chi(m) e^{2\pi i k(m+r)z} - 2\pi i \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \chi(m) e^{2\pi i k(m-r)z} \\ &= -4\pi i \sum_{k=1}^{\infty} \cos(2\pi k r z) \sum_{j=0}^{3} \chi(j) \sum_{m=0}^{\infty} e^{2\pi i k(4m+j)z} \\ &= -2\pi i \sum_{k=1}^{\infty} \frac{\cos(2\pi k r z)}{\cosh(2\pi i k z)} . \end{aligned} \tag{11.4}$$

Lastly, we need to calculate $\mathscr{L}_+(1, \chi, r)$, where

$$\mathscr{L}_+(s, \chi, r) = L(s, \chi, r) - e^{\pi i s} L(s, \chi, -r)$$

and where, for $\operatorname{Re} s > 0$ and a real,

$$L(s, \chi, a) = \sum_{k > -a} \chi(k)(k+a)^{-s} .$$

Also define, for a, x real and $\operatorname{Re} s > 1$,

$$L(s, x, a, \chi) = \sum_{k=0}^{\infty}{}' e^{\pi i k x/2} \chi(k)(k+a)^{-s} ,$$

where the prime $'$ on the summation sign indicates that the possible term $k = -a$ is omitted from the summation. The functions $L(s, \chi, a)$,

$\mathscr{L}_+(s, \chi, a)$, and $L(s, x, a, \chi)$ possess analytic continuations into the entire complex s-plane. Now apply the functional equation for $L(s, x, a, \chi)$ [9, Theorem 5.1] to get, for all s,

$$L(1-s, -r, 0, \chi) = \Gamma(s)(2/\pi)^s (i/2) e^{-\pi i s/2} \mathscr{L}_+(s, \chi, r).$$

Hence,

$$\mathscr{L}_+(1, \chi, r) = \pi L(0, -r, 0, \chi). \tag{11.5}$$

Now, for Re $s > 0$,

$$\begin{aligned} L(s, -r, 0, \chi) &= \sum_{k=1}^{\infty} e^{-\pi i k r/2} \chi(k) k^{-s} \\ &= e^{-\pi i r/2} \sum_{k=0}^{\infty} e^{-2\pi i k r}(4k+1)^{-s} - e^{-3\pi i r/2} \sum_{k=0}^{\infty} e^{-2\pi i k r}(4k+3)^{-s} \\ &= e^{-\pi i r/2} 4^{-s} \phi(-r, 1/4, s) - e^{-3\pi i r/2} 4^{-s} \phi(-r, 3/4, s), \end{aligned} \tag{11.6}$$

where, for x, a real and Re $s > 1$,

$$\phi(x, a, s) = \sum_{k=0}^{\infty}{}' e^{2\pi i k x}(k+a)^{-s}$$

denotes Lerch's zeta-function. By analytic continuation, the extreme left and right sides of (11.6) are equal for all s. Now [5, p. 164],

$$\phi(x, a, 0) = \frac{i}{2} \cot(\pi x) + \frac{1}{2}. \tag{11.7}$$

Hence, from (11.5)-(11.7),

$$\begin{aligned} z^{-1} \mathscr{L}_+(1, \chi, r) &= \frac{\pi}{z}\left(e^{-\pi i r/2} \left\{ \frac{1}{2} - \frac{i}{2} \cot(\pi r) \right\} - e^{-3\pi i r/2} \left\{ \frac{1}{2} - \frac{i}{2} \cot(\pi r) \right\} \right) \\ &= \frac{\pi}{z} e^{-\pi i r} \sin(\pi r/2) \{ \cot(\pi r) + i \} = \frac{\pi}{2z} \sec(\pi r/2). \end{aligned} \tag{11.8}$$

Substitute (11.2), (11.3), (11.4), and (11.8) into equation (4.6) of [10] to get

$$\frac{2\pi}{z} \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\pi k r/2)}{e^{\pi i k/(2z)} - 1} + \frac{\pi}{2z} \sec(\pi r/2)$$
$$= -2\pi i \sum_{k=1}^{\infty} \frac{\cos(2\pi k r z)}{\cosh(2\pi i k z)} - \pi i,$$

where $\operatorname{Im} z > 0$ and $0 < -r < 1$. Now let $z = i\pi/(2\alpha^2)$ and $r = 2n/\beta$, where $\alpha\beta = \pi$. Thus, $0 < -n < \beta/2$. Hence,

$$-4\alpha^2 i \sum_{k=1}^{\infty} \chi(k) \frac{\cos(\alpha nk)}{e^{\alpha^2 k} - 1} - \alpha^2 i \sec(\alpha n)$$

$$= -2\pi i \sum_{k=1}^{\infty} \frac{\cosh(2\beta nk)}{\cosh(\beta^2 k)} - \pi i \, .$$

Multiplying the above by $i/(4\alpha)$ yields (11.1). Now note that both sides of (11.1) are even functions of n. Thus, (11.1) is valid for $0 < |n| < \beta/2$ and, hence, by continuity, for $|n| < \beta/2$.

We remark that the differentiation of (11.1) with respect to n yields the last formula in [8] after suitable redefinitions of the parameters. However, it appears to be difficult to deduce (11.1) from the latter formula.

ENTRY 12. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi/2$, and let $0 < n < \pi/(2\alpha)$. Then

$$\alpha \sum_{k=1}^{\infty} \chi(k) \frac{\sin(\alpha nk)}{\cosh(\alpha^2 k)} = \beta \sum_{k=1}^{\infty} \chi(k) \frac{\sinh(\beta nk)}{\cosh(\beta^2 k)} \, . \tag{12.1}$$

Proof. In [12, equation (4.23)] we showed that if $0 < r < 1$ and $\alpha, \beta > 0$ with $\alpha\beta = \pi^2/16$. then

$$\sqrt{\alpha} \sum_{k=1}^{\infty} \chi(k) \frac{\sin(\pi rk/2)}{\cosh(2\alpha k)} = \sqrt{\beta} \sum_{k=1}^{\infty} \chi(k) \frac{\sinh(2\beta rk)}{\cosh(2\beta k)} \, . \tag{12.2}$$

Replace α by $\alpha^2/2$ and β by $\beta^2/2$; hence, in the new notation $\alpha\beta = \pi/2$. Let $r = 2\alpha n/\pi$. Thus, we need $0 < n < \pi/(2\alpha)$. With these substitutions, we easily find that (12.2) is transformed into (12.1).

COROLLARY OF ENTRY 12. Let $\alpha, \beta, t > 0$ with $\alpha\beta = \pi/2$ and $t = \alpha/\beta$. Let C be the positively oriented parallelogram with vertices $\pm i$ and $\pm t$. Let $\phi(z)$ be entire. For each positive integer N, define,

$$f_N(z) = \frac{\phi(4\beta N z)}{\cosh(2\pi N z) \cosh(2\pi i N z/t)} \, ,$$

and assume that $N f_N(z)$ tends to 0 boundedly on C as N tends to ∞. Then

$$(12.3) \qquad \alpha \sum_{k=1}^{\infty} \chi(k) \frac{\{\phi(\alpha k) - \phi(-\alpha k)\}}{\cosh(\alpha^2 k)}$$
$$+ i\beta \sum_{k=1}^{\infty} \chi(k) \frac{\{\phi(i\beta k) - \phi(-i\beta k)\}}{\cosh(\beta^2 k)} = 0 .$$

The above entry is not a corollary of Entry 12. In fact, as we shall see later, the converse is true. As with Entry 8iii, at the expense of brevity, the hypotheses on $f_N(z)$ can be made more explicit.

Proof. We integrate $f_N(z)$ over C. On the interior of C, $f_N(z)$ has simple poles at $z = i(2k+1)/(4N)$ and at $z = (2k+1)t/(4N)$, $-2N \leqslant k < 2N$. Straightforward calculations give

$$R\left(i\,(2k+1)/(4N)\right) = \frac{(-1)^k \phi\left(i\beta(2k+1)\right)}{2\pi i N \cosh\{(2k+1)\pi/(2t)\}}$$

and

$$R\left((2k+1)\,t/(4N)\right) = -\frac{(-1)^k t\phi\left(\beta t(2k+1)\right)}{2\pi N \cosh\{(2k+1)\pi t/2\}} .$$

Applying the residue theorem and letting N tend to ∞, we find that

$$(12.4) \qquad \lim_{N\to\infty} N \int_C f_N(z)\,dz =$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k \{\phi(i\beta(2k+1)) - \phi(-i\beta(2k+1))\}}{\cosh\{(2k+1)\pi t/2\}}$$
$$- it \sum_{k=0}^{\infty} \frac{(-1)^k \{\phi(\beta t(2k+1)) - \phi(-\beta t(2k+1))\}}{\cosh\{(2k+1)\pi t/2\}} .$$

Putting $t = \alpha/\beta$ in (12.4), we readily deduce (12.3).

Next, we show that Entry 12 is a corollary of the preceding entry. Let $\phi(z) = e^{inz}$, where $n > 0$. We see at once that (12.3) then reduces to (12.1). It is easily seen that the hypotheses on $f_N(z)$ are satisfied on the two sides of C in the upper half-plane. In the lower half-plane on C, $N f_N(z)$ tends to 0 boundedly if and only if $n < \pi/(2\alpha)$, which is precisely a hypothesis of Entry 12.

ENTRY 13. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, and let n be an integer greater than 1. Then

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k} - 1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k} - 1} = \{\alpha^n - (-\beta)^n\} \frac{B_{2n}}{4n} .$$

Entry 13 is stated without proof by Ramanujan in [60, p. 269], [61, p. 190]. The first published proof known to the author is by Rao and Ayyar [63]. Malurkar [49] and Hardy [29], [32, pp. 537-539] gave proofs shortly afterward. Later proofs were found by Nanjundiah [52], Lagrange [44], Grosswald [25], and the author [8, equation (11.10)], [12, Proposition 2.6].

COROLLARY i.

$$\sum_{k=1}^{\infty} \frac{k^5}{e^{2\pi k}-1} = \frac{1}{504}.$$

COROLLARY ii.

$$\sum_{k=1}^{\infty} \frac{k^9}{e^{2\pi k}-1} = \frac{1}{264}.$$

COROLLARY iii.

$$\sum_{k=1}^{\infty} \frac{k^{13}}{e^{2\pi k}-1} = \frac{1}{24}.$$

COROLLARY iv. If n is a positive integer, then

$$\sum_{k=1}^{\infty} \frac{k^{4n+1}}{e^{2\pi k}-1} = \frac{B_{4n+2}}{8n+4}. \tag{13.1}$$

If $\alpha = \beta = \pi$ and n is odd, then Entry 13 reduces to (13.1) if n is replaced by $2n + 1$. Corollaries i-iii are special instances of Corollary iv. Corollary iii was communicated by Ramanujan in a letter to Hardy [61, p. xxvi]. Sandham [66] also proved this special case. Aiyar [2] and Ling [48] established Corollaries i-iii. Corollary iv was proven first by Glaisher [22] in 1889. In addition to the authors who have proven Entry 13, Corollary iv has also been established by Krishnamaghari [43], Watson [73], Sandham [67], and Zucker [76].

As usual, let $\sigma_\nu(n) = \sum_{d|n} d^\nu$. It is easy to show that

$$\sum_{k=1}^{\infty} \sigma_\nu(k)\, e^{-ky} = \sum_{d=1}^{\infty} \frac{d^\nu}{e^{dy}-1}, \tag{13.2}$$

where $y > 0$. Thus, Entry 13 may be rewritten in terms of the left side of (13.2). In this form, Entry 13 perhaps was established before Hurwitz's thesis [34], [35] in 1881. Later proofs were found by Koshliakov [42], Guinand [26], and Chandrasekharan and Narasimhan [18].

ENTRY 14. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, and let n be a positive integer. Then

$$(14.1) \qquad \alpha^n \sum_{k=1}^{\infty} \chi(k) \frac{k^{2n-1}}{\cosh(\alpha k/2)} + (-\beta)^n \sum_{k=1}^{\infty} \chi(k) \frac{k^{2n-1}}{\cosh(\beta k/2)} = 0 .$$

Entry 14 has been established by Malurkar [49], Nanjundiah [52], and the author [12, Proposition 4.7].

COROLLARY OF ENTRY 14. If n is a positive integer, then

$$(14.2) \qquad \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4n-1}}{\cosh\{(2k+1)\pi/2\}} = 0 .$$

If $\alpha = \beta = \pi$ and n is even in (14.1), then (14.1) reduces to (14.2) upon the replacement of n by $2n$.

This corollary was, in fact, first established by Cauchy [17, pp. 313, 362]. Ramanujan stated (14.2) as a problem in [55]. In addition to the authors who have proved Entry 14, (14.2) has been established by Rao and Ayyar [64], Chowla [19], Sandham [67], Riesel [65], and Ling [48].

ENTRY 15. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2/4$. Then

$$(15.1) \qquad 2 \sum_{n=1}^{\infty} \chi(n) \tan^{-1}(e^{-\alpha n}) + 2 \sum_{n=1}^{\infty} \chi(n) \tan^{-1}(e^{-\beta n})$$

$$= \sum_{n=1}^{\infty} \chi(n) \frac{\operatorname{sech}(\alpha n)}{n} + \sum_{n=1}^{\infty} \chi(n) \frac{\operatorname{sech}(\beta n)}{n} = \frac{\pi}{4} .$$

Proof. A proof of the rightmost equality in (15.1) has been given by Malurkar [49], Nanjundiah [52], and the author [12, Proposition 4.5].

The leftmost equality in (15.1) follows from

$$\sum_{n=1}^{\infty} \chi(n) \frac{\operatorname{sech}(ny)}{n} = 2 \sum_{n=1}^{\infty} \frac{\chi(n)}{n} e^{-ny} \sum_{k=0}^{\infty} (-1)^k e^{-2nky}$$

$$= 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(nk)}{n} e^{-nky}$$

$$= 2 \sum_{k=1}^{\infty} \chi(k) \tan^{-1}(e^{-ky}) ,$$

where $y > 0$.

COROLLARY OF ENTRY 15. We have

$$\sum_{n=1}^{\infty} \chi(n) \tan^{-1}(e^{-\pi n/2}) = \pi/16 .$$

The corollary follows trivially from (15.1) upon setting $\alpha = \beta = \pi/2$. Rao and Ayyar [64] have also established this result. Chowla [19] has proved some formulas similar in appearance to (15.1).

ENTRY 16i. Let m and n be nonnegative integers. Then

$$\begin{aligned}\int_0^{\infty} \frac{\sin^{2n+1}x}{x} \cos^{2m}x \, dx &= \frac{\Gamma(m+1/2)\,\Gamma(n+1/2)}{2\Gamma(m+n+1)} \\ &= \int_0^{\infty} \frac{\sin^{2n+2}x}{x^2} \cos^{2m}x \, dx .\end{aligned}$$

Proof. The first equality can be found in [23, p. 457], but since the second is not in [23], we give a brief proof. (A proof of the first equality can, in fact, be given along the same lines.) Let the integral on the right side above be denoted by $I(m, n)$. We induct on m. For $m = 0$,

$$I(0, n) = \frac{\Gamma(1/2)\,\Gamma(n+1/2)}{2\Gamma(n+1)} ,$$

by [23, p. 446]. Proceeding by induction, we have

$$\begin{aligned}I(m, n) &= I(m-1, n) - I(m-1, n+1) \\ &= \frac{\Gamma(m-1/2)\,\Gamma(n+1/2)}{2\Gamma(m+n)} - \frac{\Gamma(m-1/2)\,\Gamma(n+3/2)}{2\Gamma(m+n+1)} \\ &= \frac{\Gamma(m+1/2)\,\Gamma(n+1/2)}{2\Gamma(m+n+1)} ,\end{aligned}$$

and the proof is complete.

ENTRY 16ii. Let n and p be nonnegative integers. Then

$$\begin{aligned}\int_0^{\infty} \frac{\sin^{2n+1}x}{x} \cos(2px) \, dx &= (-1)^p \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1)\,\Gamma(n+1/2)}{\Gamma(n-p+1)\,\Gamma(n+p+1)} \\ &= \int_0^{\infty} \frac{\sin^{2n+2}x}{x^2} \cos(2px) \, dx .\end{aligned}$$

Proof. We prove the first equality; the proof of the second is virtually the same. Let $I(n, p)$ denote the integral on the left side above. For $p = 0$ the proposed formula is true by Entry 16i. Thus, we assume that $p > 0$ for the remainder of the proof. We induct on n. For $n = 0$, it is easy to show that $I(0, p) = 0$ [23, p. 414], which agrees with the proposed result. Using the identities $2 \sin^2 x = 1 - \cos(2x)$ and $2 \cos(2x) \cos(2px) = \cos\{2(p+1)x\} + \cos\{2(p-1)x\}$, we find that, by the induction hypothesis,

$$
\begin{aligned}
I(n, p) &= \frac{1}{2} I(n-1, p) - \frac{1}{4} I(n-1, p+1) - \frac{1}{4} I(n-1, p-1) \\
&= \frac{(-1)^p \sqrt{\pi}}{4} \Gamma(n) \Gamma(n-1/2) \left\{ \frac{1}{\Gamma(n-p) \Gamma(n+p)} \right. \\
&\quad \left. + \frac{1}{2\Gamma(n-p-1) \Gamma(n+p+1)} + \frac{1}{2\Gamma(n-p+1) \Gamma(n+p-1)} \right\} \\
&= \frac{(-1)^p \sqrt{\pi}\, \Gamma(n+1) \Gamma(n+1/2)}{2\Gamma(n-p+1) \Gamma(n+p+1)},
\end{aligned}
$$

after several applications of the functional equation of $\Gamma(z)$.

ENTRY 17i. Let $\alpha, \beta, n > 0$ with $\alpha\beta = 2\pi$. Suppose that $\pi/(2\alpha)$ is not an integer, and let $m = [\pi/(2\alpha)]$. Let p be real. Then

$$
\begin{aligned}
(17.1) \qquad & \alpha \left\{ \frac{1}{2} + \sum_{k=1}^{m} \cos^n(\alpha k) \cos(\alpha p k) \right\} \\
&= \frac{\pi n!}{2^{n+1}} \left\{ \frac{1}{\Gamma\{\frac{1}{2}(n+p)+1\} \Gamma\{\frac{1}{2}(n-p)+1\}} \right. \\
&\quad + \sum_{k=1}^{\infty} \left(\frac{1}{\Gamma\{\frac{1}{2}(n-p+\beta k)+1\} \Gamma\{\frac{1}{2}(n+p-\beta k)+1\}} \right. \\
&\quad \left. \left. + \frac{1}{\Gamma\{\frac{1}{2}(n+p+\beta k)+1\} \Gamma\{\frac{1}{2}(n-p-\beta k)+1\}} \right) \right\}.
\end{aligned}
$$

Proof. By Stirling's formula, the right side of (17.1) converges absolutely for $n > 0$.

Apply the Poisson formula (6.1) with $f(x) = \cos^n(\alpha x) \cos(\alpha p x)$, $a = 0$, and $b = \pi/(2\alpha)$. After a simple change of variable, we find that

$$(17.2) \quad \frac{1}{2} + \sum_{k=1}^{m} \cos^n(\alpha k)\cos(\alpha pk)$$
$$= \frac{1}{\alpha}\int_0^{\pi/2} \cos^n t \cos(pt)\,dt + \frac{2}{\alpha}\sum_{k=1}^{\infty}\int_0^{\pi/2} \cos^n t\cos(pt)\cos(\beta kt)\,dt\,.$$

Now for $v > 0$ and arbitrary a [23, p. 372],

$$(17.3) \quad \int_0^{\pi/2} \cos^{v-1}x\cos(ax)\,dx$$
$$= \frac{\pi\Gamma(v+1)}{2^v v\Gamma\{{}^1/_2(v+a+1)\}\,\Gamma\{{}^1/_2(v-a+1)\}}\,.$$

If we calculate all of the integrals in (17.2) with the aid of (17.3), we arrive at (17.1) forthwith.

ENTRY 17ii. Let $\alpha, \beta, n > 0$ with $\alpha\beta = \pi/2$. Suppose that $\pi/(2\alpha)$ is not an odd integer, and let $m = [\pi/(2\alpha)]$. Let p be real. Then

$$(17.4) \quad \alpha\sum_{k=1}^{m} \chi(k)\cos^n(\alpha k)\sin(\alpha pk)$$
$$= \frac{\pi n!}{2^{2n+2}}\sum_{k=1}^{\infty}\chi(k)\left\{\frac{1}{\Gamma\{\,1/2\,(n-p+\beta k)+1\}\,\Gamma\{\,1/2\,(n+p-\beta k)+1\}}\right.$$
$$\left.-\frac{1}{\Gamma\{\,1/2\,(n+p+\beta k)+1\}\,\Gamma\{\,1/2\,(n-p-\beta k)+1\}}\right\}.$$

Proof. As before, the series on the right side of (17.4) converges absolutely for $n > 0$.

Apply the Poisson formula for sine transforms (9.2) with $f(x) = \cos^n(\alpha x)\sin(\alpha px)$, $a = 0$, and $b = \pi/(2\alpha)$. After a simple change of variable, we find that

$$(17.5) \quad \sum_{k=1}^{m} \chi(k)\cos^n(\alpha k)\sin(\alpha pk)$$
$$= \frac{1}{\alpha}\sum_{k=1}^{\infty}\chi(k)\int_0^{\pi/2}\cos^n t\sin(pt)\sin(\beta kt)\,dt\,.$$

If we calculate the integrals in (17.5) with the use of (17.3), we deduce (17.4) immediately.

COROLLARY 1 OF ENTRY 17i. Let $\alpha = \pi/(n+j)$, where n and j are positive integers of opposite parity. Let $m = [\pi/(2\alpha)]$. Then

$$\frac{1}{2} + \sum_{k=1}^{m} \cos^{2n}(\alpha k) = \frac{\sqrt{\pi}\,\Gamma(n+1/2)}{2\alpha n!}. \tag{17.6}$$

Proof. In Entry 17i replace n by $2n$ and let $p = 0$. Let $2f_n(\alpha)$ denote the infinite series on the right side of (17.1), i.e.,

$$f_n(\alpha) = \sum_{k=1}^{\infty} \frac{1}{\Gamma(n+1+k\pi/\alpha)\,\Gamma(n+1-k\pi/\alpha)}.$$

Since $f_n(\pi/(n+j)) = 0$, we see that (17.1) reduces to

$$\alpha \left\{ \frac{1}{2} + \sum_{k=1}^{m} \cos^{2n}(\alpha k) \right\} = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

which can be transformed into the desired result by the use of Legendre's duplication formula.

In fact, Ramanujan claimed that (17.6) is valid for $0 \leqslant \alpha \leqslant \pi/(n+1)$, i.e., $f_n(\alpha) \equiv 0$, $0 \leqslant \alpha \leqslant \pi/(n+1)$, provided that $\pi/(2\alpha)$ is not an integer. (Of course, for $\alpha = 0$ the result is false.) In general, $f_n(\alpha)$ does not vanish for all α in $(0, \pi/(n+1))$, as the following counterexample shows.

Let $n = 1$ and put $f(\alpha) = f_1(\alpha)$. Let $\alpha = 2\pi/5 < \pi/2$. Then

$$\begin{aligned} f(2\pi/5) &= \sum_{k=1}^{\infty} \frac{1}{\Gamma(2+5k/2)\,\Gamma(2-5k/2)} \\ &= \sum_{k=1}^{\infty} \frac{1}{(1-(5k/2)^2)(5k/2)\,\Gamma(5k/2)\,\Gamma(1-5k/2)} \\ &= \frac{2}{5\pi} \sum_{k=1}^{\infty} \frac{\sin(5\pi k/2)}{(1-(5k/2)^2)\,k} \\ &= \frac{1}{5\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(1-\{5(2k+1)/2\}^2)(2k+1)}. \end{aligned}$$

The latter series can be evaluated by the residue theorem. Let

$$h(z) = \frac{\sec(\pi z)}{\{1-(5z)^2\}\,z},$$

which has simple poles at $z = 0, \pm 1/5$, and $(2k+1)/2$, where k is an integer. Routine calculations give

$$R(0) = 1, \quad R(1/5) = -\frac{1}{2}\sec(\pi/5) = R(-1/5),$$

and

$$R((2k+1)/2) = \frac{2(-1)^{k+1}}{\pi(1 - \{5(2k+1)/2\}^2)(2k+1)}.$$

Integrate $h(z)$ over a positively oriented square C_n with center at the origin and horizontal and vertical sides of length $2n$, where n is a positive integer. As n tends to ∞,

$$\int_{C_n} h(z)\,dz = o(1).$$

Hence, applying the residue theorem and then letting n tend to ∞, we find that

$$f(2\pi/5) = \frac{1}{10}(1 - \sec(\pi/5)) \neq 0,$$

which disproves Ramanujan's claim.

Corollary 2 of Entry 17i. Let $\alpha = \pi/(n-j)$, where n and j are integers of opposite parity such that $n > 0$ and $0 \leqslant j \leqslant (n-1)/2$. Let $m = [\pi/(2\alpha)]$. Then

$$\alpha\left\{\frac{1}{2} + \sum_{k=1}^{m} \cos^{2n}(\alpha k)\right\} = \frac{\sqrt{\pi}\,\Gamma(n+1/2)}{2n!}\left\{1 + \frac{2(n!)^2}{\Gamma(n+1+\pi/\alpha)\,\Gamma(n+1-\pi/\alpha)}\right\}. \tag{17.7}$$

Proof. In Entry 17i replace n by $2n$ and let $p = 0$. After some manipulation, we find that

$$\alpha\left\{\frac{1}{2} + \sum_{k=1}^{m} \cos^{2n}(\alpha k)\right\} = \frac{\sqrt{\pi}\,\Gamma(n+1/2)}{2n!}\left\{1 + \frac{2(n!)^2}{\Gamma(n+1+\pi/\alpha)\,\Gamma(n+1-\pi/\alpha)} + 2(n!)^2 g_n(\alpha)\right\},$$

where

$$g_n(\alpha) = \sum_{k=2}^{\infty} \frac{1}{\Gamma(n+1+k\pi/\alpha)\,\Gamma(n+1-k\pi/\alpha)}.$$

For $\alpha = \pi/(n-j)$, $0 \leqslant j \leqslant (n-1)/2$, $g_n(\alpha) = 0$, and so the proof is complete.

Ramanujan, in fact, claimed that (17.7) is true for $\pi/n \leqslant \alpha \leqslant 2\pi/(n+1)$, i.e., $g_n(\alpha) \equiv 0$, $\pi/n \leqslant \alpha \leqslant 2\pi/(n+1)$, provided that $\pi/(2\alpha)$ is not an integer. Again, this claim is false, in general, and we give a counterexample.

Let $n = 3$ and put $\alpha = 2\pi/5$; so $\pi/3 < \alpha < \pi/2$. Then

$$\begin{aligned} g_3(2\pi/5) &= \sum_{k=2}^{\infty} \frac{1}{\Gamma(4+5k/2)\,\Gamma(4-5k/2)} \\ &= \frac{2}{5\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{P((2k+1)/2)(2k+1)}, \end{aligned}$$

where $P(z) = (9-25z^2)(4-25z^2)(1-25z^2)$. This series can be evaluated by the same method as used in the previous counterexample. Accordingly, we find that

$$g_3(2\pi/5) = \frac{1}{10}\left\{\frac{4-\sqrt{5}}{45} - \frac{4^4}{2079\pi}\right\} \neq 0,$$

which disproves Ramanujan's claim.

ENTRY 18. Let a_n, b_n, p_n, q_n, P_n, and Q_n be complex numbers with $a_n b_n \neq 0$. Let x and y be complex variables with $xy \neq 0$. Let

$$\phi(x) = \sum_n \frac{P_n}{p_n - a_n x} \quad \text{and} \quad \psi(y) = \sum_n \frac{Q_n}{q_n - b_n y}.$$

Then

$$(18.1) \qquad \phi(x)\,\psi(y) = \sum_n \frac{P_n}{p_n - a_n x}\,\psi\left(\frac{p_n y}{a_n x}\right) + \sum_n \frac{Q_n}{q_n - b_n y}\,\phi\left(\frac{q_n x}{b_n y}\right),$$

where it is assumed that at least one of the two double series on the right side of (18.1) converges absolutely.

Proof. Without loss of generality, assume that the latter double series on the right side of (18.1) converges absolutely. Inverting the order of summation below by absolute convergence, we have

$$\sum_n \frac{P_n}{p_n - a_n x}\,\psi\left(\frac{p_n y}{a_n x}\right) + \sum_k \frac{Q_k}{q_k - b_k y}\,\phi\left(\frac{q_k x}{b_k y}\right)$$

$$= \sum_n \frac{P_n}{p_n - a_n x} \sum_k \frac{Q_k a_n x}{a_n q_k x - b_k p_n y} + \sum_k \frac{Q_k}{q_k - b_k y} \sum_n \frac{P_n b_k y}{b_k p_n y - a_n q_k x}$$

$$= \sum_n \frac{P_n}{p_n - a_n x} \sum_k \left\{ \frac{Q_k a_n x}{a_n q_k x - b_k p_n y} + \frac{Q_k b_k y\,(p_n - a_n x)}{(q_k - b_k y)\,(b_k p_n y - a_n q_k x)} \right\}$$

$$= \sum_n \frac{P_n}{p_n - a_n x} \sum_k \frac{Q_k}{q_k - b_k y} = \phi(x)\,\psi(y)\,.$$

Despite the simplicity of the above result, Ramanujan found many interesting applications of it, as we shall see in the sequel. However, each of the following corollaries may be alternatively established by using partial fraction decompositions directly and not employing Entry 18. The following entries are valid except for obvious singularities which we shall not state.

COROLLARY 1 OF ENTRY 18. Let θ and ϕ be real with $|\theta|, |\phi| < \pi$. Then for n, x, and y complex, with x/y not purely imaginary,

$$\pi^2 n^2 xy \frac{\cos(\theta n x)\cosh(\phi n y)}{\sin(\pi n x)\ \sinh(\pi n y)}$$

$$= 1 + 2\pi n^2 xy \sum_{k=1}^{\infty} \frac{(-1)^k k \cos(k\phi)\cosh(k\theta x/y)}{(k^2 + n^2 y^2)\sinh(\pi k x/y)}$$

$$- 2\pi n^2 xy \sum_{k=1}^{\infty} \frac{(-1)^k k \cos(k\theta)\cosh(k\phi y/x)}{(k^2 - n^2 x^2)\sinh(\pi k y/x)}\,.$$

Proof. For $|\theta| \leqslant \pi$ [41, p. 377],

$$\frac{\pi n x \cos(\theta n x)}{\sin(\pi n x)} = 1 + n^2 x^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k \cos(k\theta)}{k(nx - k)}\,.$$

Similarly, for $|\phi| \leqslant \pi$,

$$\frac{\pi n y \cosh(\phi n y)}{\sinh(\pi n y)} = \frac{i\pi n y \cos(i\phi n y)}{\sin(i\pi n y)} = 1 - n^2 y^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k \cos(k\phi)}{k(iny - k)}\,.$$

Define the functions ϕ, f, ψ, and g by

$$\phi(x) = \frac{\pi n x \cos(\theta n x)}{\sin(\pi n x)} - 1 = f(x) - 1$$

and

$$\psi(y) = \frac{\pi n y \cosh(\phi n y)}{\sinh(\pi n y)} - 1 = g(y) - 1 .$$

Thus, in the notation of Entry 18, $P_k = n^2 x^2 (-1)^k \cos(k\theta)$, $p_k = -k^2$, $a_k = -kn$, $Q_k = -n^2 y^2 (-1)^k \cos(k\phi)$, $q_k = -k^2$, and $b_k = -ikn$. Applying Entry 18, we find that, for $|\theta|$, $|\phi| < \pi$ and y/x not purely imaginary,

$$\begin{aligned}
\phi(x)\psi(y) &= \frac{\pi^2 n^2 xy \cos(\theta n x)\cosh(\phi n y)}{\sin(\pi n x)\sinh(\pi n y)} - f(x) - g(y) + 1 \\
&= -f(x) + 1 - g(y) + 1 \\
&+ \pi n^2 xy \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k k \cos(k\theta)\cosh(k\phi y/x)}{(knx - k^2)\sinh(\pi k y/x)} \\
&- \pi n^2 xy \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k k \cos(k\phi)\cosh(k\theta x/y)}{(kiny - k^2)\sinh(\pi k x/y)} ,
\end{aligned}$$

which yields the desired result after some simplification.

Corollary 2 of Entry 18. Let θ and ϕ be real with $|\theta|$, $|\phi| \leqslant \pi/2$. Let n, x, and y be complex with y/x not purely imaginary. Then

(18.2)
$$\begin{aligned}
&\frac{\pi \sin(\theta n x)\sinh(\phi n y)}{4n^2 \cos(\pi n x/2)\cosh(\pi n y/2)} \\
&= y^2 \sum_{k=1}^{\infty} \frac{\chi(k)\sin(k\phi)\sinh(k\theta x/y)}{k(k^2 + n^2 y^2)\cosh(\pi k x/(2y))} \\
&+ x^2 \sum_{k=1}^{\infty} \frac{\chi(k)\sin(k\theta)\sinh(k\phi y/x)}{k(k^2 - n^2 x^2)\cosh(\pi k y/(2x))} .
\end{aligned}$$

Proof. The set of functions $\sin\{(2k+1)\theta\}$, $0 \leqslant k \leqslant \infty$, is orthogonal and complete on $[-\pi/2, \pi/2]$. An elementary calculation gives the Fourier series of $\sin(\theta n x)$ with respect to this orthogonal set. Accordingly, we find that, for $|\theta| < \pi/2$,

$$\phi(x) \equiv \frac{\sin(\theta n x)}{x \cos(\pi n x/2)} = \frac{2}{\pi x} \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1} \sin\{(2k+1)\theta\}}{nx + 2k + 1} .$$

Similarly, for $|\phi| < \pi/2$,

$$\phi(y) \equiv \frac{\sinh(\phi n y)}{y \cosh(\pi n y/2)} = \frac{\sin(i\phi n y)}{iy \cos(i\pi n y/2)}$$
$$= \frac{2i}{\pi y} \sum_{k=-\infty}^{\infty} \frac{(-1)^k \sin\{(2k+1)\phi\}}{iny + 2k + 1}.$$

Apply Entry 18 to $\phi(x)$ and $\psi(y)$ as defined above. Then $P_k = \frac{2}{\pi x}(-1)^{k+1}$ $\sin\{(2k+1)\theta\}$, $p_k = 2k+1$, $a_k = -n$, $Q_k = \frac{2i}{\pi y}(-1)^k \sin\{(2k+1)\phi\}$, $q_k = 2k+1$, and $b_k = -in$. A straightforward application of Entry 18 yields (18.2) for $|\theta|$, $|\phi| < \pi/2$. By continuity, (18.2) holds for $|\theta|$, $|\phi| \leqslant \pi/2$.

Corollary 3 of Entry 18. Let θ and ϕ be real with $|\theta|, |\phi| \leqslant \pi/2$. Let n, x, and y be complex with y/x not purely imaginary. Then

$$(18.3) \qquad \frac{\pi \cos(\theta n x) \sinh(\phi n y)}{4 \sin(\pi n x/2) \cosh(\pi n y/2)} = \frac{\phi y}{2x}$$
$$+ n^2 y^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \sin\{(2k+1)\phi\} \cosh\{(2k+1)\theta x/y\}}{(2k+1)\{(2k+1)^2 + n^2 y^2\} \sinh\{(2k+1)\pi x/(2y)\}}$$
$$+ n^2 x^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k\theta) \sinh(2k\phi y/x)}{2k\{(2k)^2 - n^2 x^2\} \cosh(\pi k y/x)}.$$

Proof. We first calculate the Fourier series of $\cos(\theta n x)$ with respect to the complete orthogonal set $\cos(2k\theta)$, $0 \leqslant k < \infty$, on $[-\pi/2, \pi/2]$. Accordingly, we find that

$$\frac{\cos(\theta n x)}{x \sin(\pi n x/2)} = \frac{2}{\pi x} \sum_{k=-\infty}^{\infty} \frac{(-1)^k \cos(2k\theta)}{nx + 2k}.$$

Define $\phi(x) = \frac{\cos(\theta n x)}{x \sin(\pi n x/2)} - g(x)$, where $g(x) = 2/(\pi n x^2)$. Thus, in the notation of Entry 18, $P_k = \frac{2}{\pi x}(-1)^k \cos(2k\theta)$, $p_k = 2k$, and $a_k = -n$, where $k \neq 0$.

Let $\psi(y)$ be as in the previous corollary. Thus, by Entry 18, for $|\theta|$, $|\phi| < \pi/2$ and y/x not purely imaginary,

$$\frac{\cos(\theta n x)\sinh(\phi n y)}{xy\sin(\pi n x/2)\cosh(\pi n y/2)} - \psi(y)\,g(x) \tag{18.4}$$

$$= f(x,y) + \frac{n}{\pi y}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{(-1)^k\cos(2k\theta)\sinh(2k\phi y/x)}{k(nx+2k)\cosh(\pi k y/x)}$$

$$+\frac{2in}{\pi x}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}\sin\{(2k+1)\phi\}\cosh\{(2k+1)\theta x/y\}}{(2k+1)(iny+2k+1)\sinh\{(2k+1)\pi x/(2y)\}},$$

where

$$f(x,y) = \frac{4iny}{\pi^2x^2}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin\{(2k+1)\phi\}}{(iny+2k+1)(2k+1)^2} \tag{18.5}$$

$$= -g(x)\psi(y) + \frac{4iny}{\pi^2x^2}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin\{(2k+1)\phi\}}{iny+2k+1}$$

$$\cdot\left\{\frac{1}{(2k+1)^2}+\frac{1}{n^2y^2}\right\}$$

$$= -g(x)\psi(y) + \frac{8}{\pi^2x^2}\sum_{k=0}^{\infty}\frac{(-1)^k\sin\{(2k+1)\phi\}}{(2k+1)^2}$$

$$= -g(x)\psi(y) + \frac{2\phi}{\pi x^2}.$$

In this last step, we have used the Fourier series of ϕ with respect to the complete orthogonal set $\sin\{(2k+1)\phi\}$, $0 \leqslant k < \infty$, on $[-\pi/2, \pi/2]$. If we substitute (18.5) into (18.4), we obtain (18.3) for $|\theta|$, $|\phi| < \pi/2$ after some simplification. By continuity, (18.3) is valid for $|\theta|$, $|\phi| \leqslant \pi/2$.

ENTRY 19i. We have

$$\pi^2 xy\cot(\pi x)\coth(\pi y) = 1 \tag{19.1}$$

$$+ 2\pi xy\sum_{n=1}^{\infty}\frac{n\coth(\pi n x/y)}{n^2+y^2} - 2\pi xy\sum_{n=1}^{\infty}\frac{n\coth(\pi n y/x)}{n^2-x^2}.$$

We have stated Entry 19i with no hypotheses because, in general, the two series on the right side of (19.1) do not converge. Ramanujan evidently used Entry 18 to derive Entry 19i, and so we formally derive Entry 19i in this way. From (1.9), we have

$$\pi x\cot(\pi x) = 1 + x^2\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{nx-n^2} \tag{19.2}$$

and

$$\pi y \coth(\pi y) = 1 + y^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2 - iny}.$$

Apply Entry 18 to $\phi(x) = \pi x \cot(\pi x) - 1$ and $\psi(y) = \pi y \coth(\pi y) - 1$. Ignoring the fact that the resulting two series on the right side of (18.1) diverge, we arrive at (19.1) quite easily.

ENTRY 19ii. Let x and y be complex numbers such that x/y is not purely imaginary. Then

$$\pi^2 xy \csc(\pi x) \operatorname{csch}(\pi y) = 1$$

$$+ 2\pi xy \sum_{n=1}^{\infty} \frac{(-1)^n n \operatorname{csch}(\pi n x/y)}{n^2 + y^2} - 2\pi xy \sum_{n=1}^{\infty} \frac{(-1)^n n \operatorname{csch}(\pi n y/x)}{n^2 - x^2}.$$

Proof. From [75, p. 136],

$$\phi(x) \equiv \pi x \csc(\pi x) - 1 = x^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{nx - n^2}$$

and

$$\psi(y) \equiv \pi y \operatorname{csch}(\pi y) - 1 = y^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2 - inx}.$$

Apply Entry 18 with $\phi(x)$ and $\psi(y)$ as defined above. Thus, $P_n = (-1)^n x^2$, $p_n = -n^2$, $a_n = -n$, $Q_n = (-1)^n y^2$, $q_n = n^2$, and $b_n = in$. Hence,

$$\phi(x)\psi(y) = x^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{nx - n^2} \left\{ \frac{\pi n y}{x} \operatorname{csch}\left(\frac{\pi n y}{x}\right) - 1 \right\}$$

$$+ y^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2 - iny} \left\{ \frac{\pi i n x}{y} \csc\left(\frac{\pi i n x}{y}\right) - 1 \right\}.$$

The completion of the proof is straightforward, and we omit it.

ENTRY 19iii. Let x and y be complex numbers such that y/x is not purely imaginary. Then

$$\frac{\pi}{4} \tan(\pi x/2) \tanh(\pi y/2)$$

$$= y^2 \sum_{n=0}^{\infty} \frac{\tanh\{(2n+1)\pi x/(2y)\}}{(2n+1)\{(2n+1)^2 + y^2\}} + x^2 \sum_{n=0}^{\infty} \frac{\tanh\{(2n+1)\pi y/(2x)\}}{(2n+1)\{(2n+1)^2 - x^2\}}.$$

Proof. From [23, p. 36],

$$\phi(x) \equiv \frac{1}{x}\tan(\pi x/2) = -\frac{2}{\pi x}\sum_{n=-\infty}^{\infty}{}' \frac{1}{2n+1+x}$$

and

$$\psi(y) \equiv \frac{1}{y}\tanh(\pi y/2) = \frac{2i}{\pi y}\sum_{n=-\infty}^{\infty}{}' \frac{1}{2n+1+iy},$$

where the prime $'$ on the summation sign on each right side above indicates that the sum is to be interpreted as $\lim_{N\to\infty}\sum_{n=-N}^{N}$. Apply Entry 18 to $\phi(x)$ and $\psi(y)$ as defined above. Thus, $P_n = -2/(\pi x)$, $p_n = 2n+1$, $a_n = -1$, $Q_n = 2i/(\pi y)$, $q_n = 2n+1$, and $b_n = -i$. Hence,

$$\phi(x)\psi(y) = -\frac{2}{\pi y}\sum_{n=-\infty}^{\infty}\frac{\tanh\{(2n+1)\pi y/(2x)\}}{(2n+1)(2n+1+x)}$$
$$+\frac{2i}{\pi x}\sum_{n=-\infty}^{\infty}\frac{\tanh\{(2n+1)\pi x/(2y)\}}{(2n+1)(2n+1+iy)},$$

and, after a little simplification, the desired result follows.

ENTRY 19iv. Let x and y be complex numbers such that y/x is not purely imaginary. Then

$$\frac{\pi}{4}\sec(\pi x/2)\operatorname{sech}(\pi y/2)$$
$$= \sum_{n=1}^{\infty}\frac{\chi(n)\, n \operatorname{sech}\{\pi n x/(2y)\}}{n^2+y^2} + \sum_{n=1}^{\infty}\frac{\chi(n)\, n\operatorname{sech}\{\pi n y/(2x)\}}{n^2-x^2}.$$

Proof. From (1.2),

$$\phi(x) \equiv \sec(\pi x/2) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n+1+x}$$

and

$$\psi(y) \equiv \operatorname{sech}(\pi y/2) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n+1+iy}. \tag{19.3}$$

Apply Entry 18 with ϕ and ψ defined as above, and we readily obtain the desired result.

ENTRY 19v. Let x and y be complex numbers such that y/x is not purely imaginary. Then

$$\frac{\pi}{4}\cot(\pi x/2)\operatorname{sech}(\pi y/2) = \frac{1}{2x}$$
$$- y\sum_{n=1}^{\infty}\frac{\chi(n)\coth\{\pi n x/(2y)\}}{n^2+y^2} - x\sum_{n=1}^{\infty}\frac{\operatorname{sech}(\pi n y/x)}{(2n)^2-x^2}.$$

Proof. From (19.2),

$$\phi(x) \equiv \cot(\pi x/2) - \frac{2}{\pi x} = \frac{x}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{nx/2-n^2}.$$

Apply Entry 18 to $\phi(x)$ given above and to $\psi(y)$ given by (19.3). Hence,

$$\phi(x)\psi(y) = \frac{x}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\operatorname{sech}(\pi n y/x)}{nx/2-n^2}$$
$$+ \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n+1+iy}\left\{\cot\left(\frac{\pi i(2n+1)x}{2y}\right) + \frac{2iy}{(2n+1)\pi x}\right\}$$
$$= -\frac{4x}{\pi}\sum_{n=1}^{\infty}\frac{\operatorname{sech}(\pi n y/x)}{(2n)^2-x^2} - \frac{2}{\pi x}\operatorname{sech}(\pi y/2)$$
$$- \frac{4y}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\coth\{(2n+1)\pi x/(2y)\}}{(2n+1)^2+y^2}$$
$$+ \frac{4}{\pi^2 x}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n+1+iy}\left\{1+\frac{iy}{2n+1}\right\}.$$

The last series above reduces to twice Gregory's series for $\pi/4$. Hence, after a little simplification, the above reduces to the desired result.

After Entry 19v, Ramanujan remarks that similar formulas can be derived for $\tan(\pi x/2)\coth(\pi y/2)$ and $\sec(\pi x/2)\coth(\pi y/2)$.

ENTRY 20i. We have

$$\pi^2 z^2\cot(\pi z)\coth(\pi z) = 1 - 4\pi z^4\sum_{n=1}^{\infty}\frac{n\coth(\pi n)}{n^4-z^4}.$$

Note that if we set $x = y = z$ in (19.1), we obtain the above equality. However, as previously observed, the two series on the right side of (19.1) do not converge for $x = y$. Nonetheless, if we calculate the partial fraction expansion of $\pi^2 \cot(\pi z) \coth(\pi z)/z$, we readily deduce Entry 20i. Because the calculation is quite routine, we omit it.

COROLLARY OF ENTRY 20i. We have

$$\pi^2 z^2 \frac{\cosh(\pi z\sqrt{2}) + \cos(\pi z\sqrt{2})}{\cosh(\pi z\sqrt{2}) - \cos(\pi z\sqrt{2})} = 1 + 4\pi z^4 \sum_{n=1}^{\infty} \frac{n \coth(\pi n)}{n^4 + z^4}.$$

Proof. In Entry 20i replace z by $e^{\pi i/4}z$. We see that we must calculate

$$i \cot(\pi e^{\pi i/4} z) \coth(\pi e^{\pi i/4} z)$$
$$= \frac{\cosh(\pi z(1-i)/\sqrt{2}) \cosh(\pi z(1+i)/\sqrt{2})}{\sinh(\pi z(1-i)/\sqrt{2}) \sinh(\pi z(1+i)/\sqrt{2})}$$
$$= \frac{\cosh(\pi z\sqrt{2}) + \cos(\pi z\sqrt{2})}{\cosh(\pi z\sqrt{2}) - \cos(\pi z\sqrt{2})}.$$

The desired equality now follows.

ENTRY 20ii. We have

$$\pi^2 z^2 \csc(\pi z) \operatorname{csch}(\pi z) = 1 - 4\pi z^4 \sum_{n=1}^{\infty} \frac{(-1)^n n \operatorname{csch}(\pi n)}{n^4 - z^4}.$$

Proof. Let $x = y = z$ in Entry 19ii, and the result follows.

COROLLARY OF ENTRY 20iii. We have

$$\frac{2\pi^2 z^2}{\cosh(\pi z\sqrt{2}) - \cos(\pi z\sqrt{2})} = 1 + 4\pi z^4 \sum_{n=1}^{\infty} \frac{(-1)^n n \operatorname{csch}(\pi n)}{n^4 + z^4}.$$

Proof. In Entry 20ii replace z be $e^{\pi i/4}z$. Use part of the calculation in the proof of the Corollary of Entry 20i, and the desired result easily follows.

ENTRY 20iii. We have

$$\frac{\pi}{8z^2} \tan(\pi z/2) \tanh(\pi z/2) = \sum_{n=0}^{\infty} \frac{(2n+1) \tanh\{(2n+1)\pi/2\}}{(2n+1)^4 - z^4}.$$

Proof. Put $x = y = z$ in Entry 19iii, and the result readily follows.

COROLLARY OF ENTRY 20iii. We have

$$\frac{\pi}{8z^2} \frac{\cosh(\pi z/\sqrt{2}) - \cos(\pi z/\sqrt{2})}{\cosh(\pi z/\sqrt{2}) + \cos(\pi z/\sqrt{2})}$$
$$= \sum_{n=0}^{\infty} \frac{(2n+1)\tanh\{(2n+1)\pi/2\}}{(2n+1)^4 + z^4}.$$

Proof. Replace z by $e^{\pi i/4} z$ in Entry 20iii. The calculation that is needed is precisely of the same type as that given in the proof of the Corollary of Entry 20i.

ENTRY 20iv. We have

$$\frac{\pi}{8} \sec(\pi z/2) \operatorname{sech}(\pi z/2) = \sum_{n=1}^{\infty} \chi(n) \frac{n^3 \operatorname{sech}(\pi n/2)}{n^4 - z^4}.$$

Proof. Let $x = y = z$ in Entry 19iv, and the result follows forthwith.

COROLLARY OF ENTRY 20iv. We have

$$\frac{\pi/4}{\cosh(\pi z/\sqrt{2}) + \cos(\pi z/\sqrt{2})} = \sum_{n=1}^{\infty} \chi(n) \frac{n^3 \operatorname{sech}(\pi n/2)}{n^4 + z^4}.$$

Proof. The corollary follows from Entry 20iv upon the replacement of z by $e^{\pi i/4} z$ and from the calculation in the proof of Entry 20i.

ENTRY 21i. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, and let n be any nonzero integer. Then

$$\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\alpha k} - 1} \right\}$$
$$= (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\beta k} - 1} \right\}$$
$$- 2^{2n} \sum_{k=0}^{n+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2n+2-2k}}{(2n+2-2k)!} \alpha^{n+1-k} \beta^k,$$

where B_j denotes the jth Bernoulli number.

Entry 21i is perhaps the most well-known result in Chapter 14. For $\alpha = \beta = \pi$ and n odd and positive, the theorem is first due to Lerch [45]. A proof of the more general Entry 21i was first given by Malurkar [49]. Other proofs of the aforementioned special case or of the full result have been given by Grosswald [24], [25], Smart [71], Katayama [37], [40], Riesel [57], and the author [11], [12]. Several other authors have established transformation formulas from which Entry 21i readily follows. Thus, although Entry 21i was not explicitly stated by them, Guinand [26], [27], Apostol [4], Mikolás [51], Iseki [36], Chandrasekharan and Narasimhan [18], Glaeske [20], [21], Bodendiek [15], and Bodendiek and Halbritter [16] have essentially proved Entry 21i. For a more detailed discussion of this formula, see the author's expository paper [7]. Lastly, note that for $n < -1$, Entry 21i yields Entry 13 (with n replaced by $-n$).

ENTRY 21ii. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2/4$. Let n be any integer. Then

$$\alpha^{-n} \sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\alpha k)}{k^{2n+1}} + (-\beta)^{-n} \sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\beta k)}{k^{2n+1}}$$

$$= \frac{\pi}{4} \sum_{k=0}^{n} (-1)^k \frac{E_{2k}}{(2k)!} \frac{E_{2n-2k}}{(2n-2k)!} \alpha^{n-k}\beta^k ,$$

where E_j denotes the jth Euler number.

Note that the latter equality in Entry 15 is the case $n = 0$ of Entry 21ii. Also observe that Entry 21ii reduces to Entry 14 when $n < 0$. (The parameters n, α, and β must be replaced by $-n$, $\alpha/2$, and $\beta/2$, respectively, to obtain Entry 14).

Proofs of Entry 21ii have been given by, firstly, Malurkar [49] and then by Nanjundiah [52] and the author [12, Proposition 4.5].

For $\operatorname{Re} s > 0$, let

$$L(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s} . \tag{21.1}$$

Note that $L(s)$ is a Dirichlet L-function and so can be analytically continued to an entire function.

ENTRY 21iii. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, and let n be any integer. Then

$$\alpha^{-n+1/2}\left\{\frac{1}{2}L(2n)+\sum_{k=1}^{\infty}\frac{\chi(k)}{k^{2n}(e^{\alpha k}-1)}\right\}$$
$$=\frac{(-1)^n\beta^{-n+1/2}}{2^{2n+1}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}\cosh(\beta k)}$$
$$+\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^k}{2^{2k}}\frac{E_{2k}}{(2k)!}\frac{B_{2n-2k}}{(2n-2k)!}\alpha^{n-k}\beta^{k+1/2}.$$

The first published proof of Entry 21iii was given by Chowla [19, equation (1.2)]. The author [12, equation (3.20)] has also given a proof. (Unfortunately, formula (3.20) contains an error; replace $(\beta/8)^k$ by $\beta^{k+\frac{1}{2}}2^{-4k}$ at the end of (3.20).) Entry 21iii also follows from results of Katayama [38], [39].

Entry 22i. Let x and y be complex numbers with y/x not purely imaginary. Then

$$\text{(22.1)}\quad \pi^2xy\,\frac{\left\{\begin{aligned}&\cosh\{\pi(x+y)\sqrt{2}\}+\cos\{\pi(x-y)\sqrt{2}\}\\&\qquad-\cosh\{\pi(x-y)\sqrt{2}\}-\cos\{\pi(x+y)\sqrt{2}\}\end{aligned}\right\}}{\{\cosh(\pi x\sqrt{2})-\cos(\pi x\sqrt{2})\}\{\cosh(\pi y\sqrt{2})-\cos(\pi y\sqrt{2})\}}$$
$$=2+4\pi xy^3\sum_{n=1}^{\infty}\frac{n\coth(\pi nx/y)}{n^4+y^4}+4\pi x^3y\sum_{n=1}^{\infty}\frac{n\coth(\pi ny/x)}{n^4+x^4}.$$

Proof. Let

$$zf(z)=\pi^2\cot(\pi zx)\coth(\pi zy)\quad\text{and}\quad zg(z)=\pi^2\cot(\pi zy)\coth(\pi zx).$$

If we expand $f(z)$ and $g(z)$ into partial fractions, we obtain

$$xy\{f(z)+g(z)\}=\frac{2}{z^3}$$
$$+4\pi x^3yz\sum_{n=1}^{\infty}\frac{n\coth(\pi ny/x)}{x^4z^4-n^4}+4\pi xy^3z\sum_{n=1}^{\infty}\frac{n\coth(\pi nx/y)}{y^4z^4-n^4}.$$

If $z=1$, the above becomes

$$\text{(22.2)}\quad \pi^2xy\{\cot(\pi x)\coth(\pi y)+\cot(\pi y)\coth(\pi x)\}=2$$
$$-4\pi x^3y\sum_{n=1}^{\infty}\frac{n\coth(\pi ny/x)}{n^4-x^4}-4\pi xy^3\sum_{n=1}^{\infty}\frac{n\coth(\pi nx/y)}{n^4-y^4}.$$

Replace x by $e^{\pi i/4}x$ and y by $e^{\pi i/4}y$ in the above. The right side of (22.2) then becomes the right side of (22.1). On the left side of (22.2) we have

$$(22.3) \qquad \pi^2 xy \left\{ \frac{\cosh(a-ia)\cosh(b+ib)}{\sinh(a-ia)\,\sinh(b+ib)} + \frac{\cosh(b-ib)\cosh(a+ia)}{\sinh(b-ib)\,\sinh(a+ia)} \right\}$$
$$= \frac{\pi^2 xy\{F(a,b) + F(b,a)\}}{G(a,b)},$$

where $a = \pi x/\sqrt{2}$, $b = \pi y/\sqrt{2}$,

$$F(a,b) = \cosh(a-ia)\sinh(a+ia)\cosh(b+ib)\sinh(b-ib),$$

and

$$G(a,b) = \sinh(a-ia)\sinh(a+ia)\sinh(b-ib)\sinh(b+ib).$$

Now,

$$F(a,b) = \frac{1}{4}\{\sinh(2a) + i\sin(2a)\}\{\sinh(2b) - i\sin(2b)\},$$

and so

$$(22.4) \qquad F(a,b) + F(b,a) = \frac{1}{2}\{\sinh(2a)\sinh(2b) + \sin(2a)\sin(2b)\}$$
$$= \frac{1}{4}(\cosh\{2(a+b)\} - \cosh\{2(a-b)\} + \cos\{2(a-b)\}$$
$$- \cos\{2(a+b)\}).$$

Also,

$$(22.5) \qquad G(a,b) = \frac{1}{4}\{\cosh(2a) - \cos(2a)\}\{\cosh(2b) - \cos(2b)\}.$$

If we substitute (22.4) and (22.5) into (22.3), we find that (22.3) is transformed into the left side of (22.1). This completes the proof.

Entry 22i in the Notebook is slightly in error. Ramanujan has replaced the numerator of the left side of (22.1) by

$$\cosh\{\pi(x+y)\sqrt{2}\}\cos\{\pi(x-y)\sqrt{2}\}$$
$$-\cosh\{\pi(x-y)\sqrt{2}\}\cos\{\pi(x+y)\sqrt{2}\}.$$

It also may be remarked that *formally* (22.2) can be derived from Entry 19i.

ENTRY 22ii. Let $n \geqslant 0$. Then

$$(22.6) \qquad \int_0^\infty \frac{\cos(2nx)\,dx}{\cosh(\pi\sqrt{x}) + \cos(\pi\sqrt{x})} = 2\sum_{k=1}^{\infty} \frac{\chi(k)\,k}{\cosh(\pi k/2)} e^{-nk^2}.$$

Proof. Let

$$f(z) = \frac{1}{\cosh(\pi\sqrt{z}) + \cos(\pi\sqrt{z})}.$$

We expand f into its partial fraction decomposition. There are simple poles at $z = \pm i(2k+1)^2/2$, $-\infty < k < \infty$. Since

$$R\left(i(2k+1)^2/2\right) = \frac{(-1)^k(2k+1)}{\pi i \cosh\{(2k+1)\pi/2\}} = -R\left(-i(2k+1)^2/2\right),$$

we readily find that

$$(22.7) \qquad f(z) = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^3}{\cosh\{(2k+1)\pi/2\}\left(z^2+(2k+1)^4/4\right)} + g(z),$$

where $g(z)$ is entire. By the same argument as that used in the proof of Entry 4, $g(z) \equiv 0$.

Letting $z = x$, we multiply both sides of (22.7) by $\cos(2nx)$ and integrate with respect to x over $[0, \infty)$. Inverting the order of integration and summation by absolute convergence and using [23, p. 406]

$$\int_0^\infty \frac{\cos(ax)\,dx}{x^2+b^2} = \frac{\pi}{2b}e^{-ab}, \qquad a \geqslant 0, \; b > 0,$$

we find that

$$\int_0^\infty f(x)\cos(2nx)\,dx = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\chi(k)k^3}{\cosh(\pi k/2)}\int_0^\infty \frac{\cos(2nx)\,dx}{x^2+k^4/4}$$
$$= 2\sum_{k=1}^{\infty}\frac{\chi(k)k}{\cosh(\pi k/2)}e^{-nk^2},$$

which completes the proof.

The factor 2 on the right side of (22.6) is missing in the Notebook. The integral evaluated in (22.6) is very similar to integrals evaluated by Ramanujan in [59], [61, pp. 59-67]. Ramanujan claimed that the next entry is a corollary of Entry 22ii. We cannot show this and so proceed from scratch.

Corollary of Entry 22ii. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^3/4$. Then

$$(22.8) \qquad \sum_{n=1}^{\infty}\frac{\chi(n)}{n\{\cosh\sqrt{\alpha n} + \cos\sqrt{\alpha n}\}} + \sum_{n=1}^{\infty}\frac{\chi(n)}{n\cosh(\pi n/2)\cosh^2(\beta n^2)} = \frac{\pi}{8}.$$

Proof. Let N be an even positive integer. We shall let N tend to ∞, but we shall further restrict N be requiring that N^2 remain at a bounded distance from the numbers $(2n+1)\,\alpha/\pi^2$, where n is a positive integer. Let

$$f_N(z) = \frac{1}{z\{\cosh(\pi N z) + \cos(\pi N z)\}\cos(2\beta N^2 z^2)}\,.$$

Elementary considerations show that $f(z)$ has simple poles at $z = 0$, at $z = (2n+1)(\pm 1 + i)/(2N)$, where n is an integer, and at $z = \pm\sqrt{(2k+1)\,\alpha/(N\pi)}$, where k is an integer. Straightforward calculations yield $R(0) = ½$,

$$R\big((2n+1)(\pm 1 + i)/(2N)\big)$$
$$= \frac{(-1)^{n+1}}{\pi(2n+1)\cosh\{(2n+1)\pi/2\}\cosh\{(2n+1)^2\beta\}}\,,$$

and

$$R\big(\pm\sqrt{(2k+1)\,\alpha/(N\pi)}\big)$$
$$= \frac{(-1)^{k+1}}{\pi(2k+1)\{\cosh\sqrt{(2k+1)\,\alpha} + \cos\sqrt{(2k+1)\,\alpha}\}}\,.$$

Let C denote the positively oriented rhombus with vertices ± 1 and $\pm i$. Hence, employing the residue theorem and letting N tend to ∞, we find that

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_C f_N(z)\,dz = \frac{1}{2} \tag{22.9}$$
$$+ \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\cosh\{(2n+1)\pi/2\}\cosh\{(2n+1)^2\beta\}}$$
$$+ \frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}}{(2k+1)\{\cosh\sqrt{(2k+1)\,\alpha} + \cos\sqrt{(2k+1)\,\alpha}\}}\,.$$

By the definition of f_N and the choice of N, it is easily seen that the limit on the left side of (22.9) is zero. A slight rearrangement of (22.9) yields (22.8), and we are done.

ENTRY 22iii. Let $\alpha, \beta > 0$ with $\alpha\beta = 4\pi^3$, and let γ denote Euler's constant. Then

$$(22.10)\qquad \frac{7\alpha}{720}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\sqrt{\alpha n}}{n(\cosh\sqrt{\alpha n}-\cos\sqrt{\alpha n})}=\frac{\gamma+\operatorname{Log}(2\pi/\beta)}{4}$$
$$+\frac{\beta}{48\pi}+\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}+\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n(e^{\beta n^2}-1)}.$$

Furthermore,

$$(22.11)\qquad \sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}=\frac{1}{4}\operatorname{Log}(4/\pi)-\frac{\pi}{12}+\operatorname{Log}\Gamma(3/4).$$

In the Notebook, formula (22.11) contains a misprint; $\operatorname{Log}\Gamma(3/4)$ is replaced by $\frac{1}{4}\operatorname{Log}\Gamma(3/4)$.

Proof. We first prove (22.11). A direct calculation gives

$$(22.12)\qquad \sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}=-\operatorname{Log}\prod_{n=1}^{\infty}(1-q^{2n}),$$

where $q=e^{-\pi}$. Now from [75, p. 488, problem 10],

$$(22.13)\qquad \prod_{n=1}^{\infty}(1-q^{2n})^6=\frac{2kk'K^3}{\pi^3 q^{1/2}},$$

where k, k', and K have their standard meanings in the theory of elliptic functions. Here, $k=k'=1/\sqrt{2}$ and $K=\pi^{3/2}/(2\Gamma^2(3/4))$. (See [76], for example.) Thus, (22.12) and (22.13) yield

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}=-\frac{1}{6}\operatorname{Log}\left\{\frac{\pi^{3/2}}{2^3\Gamma^6(3/4)}\right\}-\frac{\pi}{12}$$
$$=\frac{1}{4}\operatorname{Log}(4/\pi)+\operatorname{Log}\Gamma(3/4)-\pi/12,$$

as desired.

We now prove (22.10). Let $N=n+½$, where n is a positive integer. We shall let N tend to ∞ through a sequence such that $N^2\pi^2/\alpha$ remains at a bounded distance away from the positive integers. Let

$$f_N(z)=\frac{\coth(\pi Nz)\cot(\pi Nz)}{z(e^{\beta N^2z^2}-1)}.$$

The function $f_N(z)$ has simple poles at $z=\pm\sqrt{\alpha k}(1+i)/(2\pi N)$, at $z=ik/N$, and at $z=ik/N$, where k is a nonzero integer. In addition,

$f_N(z)$ has a quintuple pole at $z = 0$. Using elementary trigonometric identities, we find, after some calculation, that

$$(22.14) \qquad R(\pm\sqrt{\alpha k}(1+i)/(2\pi N)) = R(\pm\sqrt{-\alpha k}(1+i)/(2\pi N))$$
$$= -\frac{1}{4\pi k}\left\{\frac{2\cos\sqrt{\alpha k}}{\cosh\sqrt{\alpha k} - \cos\sqrt{\alpha k}} + 1\right\}.$$

Easier calculations yield

$$R(\pm ik/N) = -\frac{\coth(\pi k)}{\pi k(e^{-\beta k^2}-1)}$$

and

$$R(\pm k/N) = \frac{\coth(\pi k)}{\pi k(e^{\beta k^2}-1)}.$$

Observe that

$$(22.15) \qquad R(\pm ik/N) + R(\pm k/N) = \frac{2\coth(\pi k)}{\pi k(e^{\beta k^2}-1)} + \frac{\coth(\pi k)}{\pi k}.$$

To calculate the residue at $z = 0$, write

$$f_N(z) = \frac{1}{z}\left\{\frac{1}{\pi N z} + \frac{\pi N z}{3} - \frac{(\pi N z)^3}{45} + \ldots\right\}$$
$$\cdot\left\{\frac{1}{\pi N z} - \frac{\pi N z}{3} - \frac{(\pi N z)^3}{45} + \ldots\right\}\frac{1}{\beta N^2 z^2}\left\{1 - \frac{\beta N^2 z^2}{2} + \frac{(\beta N^2 z^2)^2}{12} + \ldots\right\}.$$

After some simplification, we find that

$$(22.16) \qquad R(0) = \frac{\beta}{12\pi^2} - \frac{7\alpha}{180\pi}.$$

Let C denote the positively oriented rhombus with vertices ± 1 and $\pm i$. By our choice of N, there are no poles of f_N on C. Applying the residue theorem and employing (22.14)-(22.16), we find that

$$(22.17) \qquad \frac{1}{2\pi i}\int_C f_N(z)\,dz = -\frac{2}{\pi}\sum_{1\leq k\leq \pi^2N^2/\alpha}\frac{\cos\sqrt{\alpha k}}{k(\cosh\sqrt{\alpha k} - \cos\sqrt{\alpha k})}$$
$$-\frac{1}{\pi}\sum_{1\leq k\leq \pi^2N^2/\alpha}\frac{1}{k} + \frac{4}{\pi}\sum_{1\leq k\leq N}\frac{\coth(\pi k)}{k(e^{\beta k^2}-1)}$$
$$+\frac{2}{\pi}\sum_{1\leq k\leq N}\frac{\coth(\pi k)}{k} + \frac{\beta}{12\pi^2} - \frac{7\alpha}{180\pi}.$$

Next, we calculate directly the integral on the left side of (22.17). Let C_j denote the part of C in the jth quadrant, $1 \leqslant j \leqslant 4$. On C_1 set $z = 1 - x + ix$, $0 \leqslant x \leqslant 1$, and on C_3 set $z = x - 1 - ix$, $0 \leqslant x \leqslant 1$. Then in either case,

$$\lim_{N\to\infty} f_N(z) = \begin{cases} 0, & 0 < x < 1/2, \\ i/z, & 1/2 < x < 1. \end{cases} \tag{22.18}$$

On C_2 set $z = -x + (1-x)i$, $0 \leqslant x \leqslant 1$, and on C_4 set $z = x + (x-1)i$, $0 \leqslant x \leqslant 1$. Then in either case,

$$\lim_{N\to\infty} f_N(z) = \begin{cases} -i/z, & 0 < x < 1/2, \\ 0, & 1/2 < x < 1. \end{cases} \tag{22.19}$$

By the choice of N, the convergence in (22.18) and (22.19) is bounded on C as N tends to ∞. Hence, by the bounded convergence theorem,

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_C f_N(z)\, dz \tag{22.20}$$

$$= \frac{1}{2\pi}\left\{\int_{(1+i)/2}^{i} - \int_{i}^{(-1+i)/2} + \int_{(-1-i)/2}^{-i} - \int_{-i}^{(1-i)/2}\right\}\frac{dz}{z} = \frac{1}{\pi}\operatorname{Log} 2 .$$

Returning to (22.17), we examine

$$2 \sum_{1\leqq k\leqq N} \frac{\coth(\pi k)}{k} - \sum_{1\leqq k\leqq \pi^2N^2/\alpha} \frac{1}{k} \tag{22.21}$$

$$= 4 \sum_{1\leqq k\leqq N} \frac{1}{k\,(e^{2\pi k}-1)} + 2 \sum_{1\leqq k\leqq N} \frac{1}{k} - \sum_{1\leqq k\leqq \pi^2N^2/\alpha} \frac{1}{k} .$$

Now [6, p. 43],

$$2 \sum_{1\leqq k\leqq N} \frac{1}{k} - \sum_{1\leqq k\leqq \pi^2N^2/\alpha} \frac{1}{k} \tag{22.22}$$

$$= 2\{\operatorname{Log} N + \gamma + O(1/N)\} - \{\operatorname{Log}(\pi^2N^2/\alpha) + \gamma + O(1/N^2)\}$$

$$= \gamma - 2\log\pi + \operatorname{Log}\alpha + O(1/N) .$$

Thus, letting N tend to ∞ in (22.17), using (22.20)-(22.22), and multiplying both sides by π, we deduce that

$$\operatorname{Log} 2 = -2 \sum_{k=1}^{\infty} \frac{\cos \sqrt{\alpha k}}{k(\cosh \sqrt{\alpha k} - \cos \sqrt{\alpha k})} + 4 \sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k(e^{\beta k^2} - 1)}$$

$$+ 4 \sum_{k=1}^{\infty} \frac{1}{k(e^{2\pi k} - 1)} + \gamma - 2 \operatorname{Log} \pi + \operatorname{Log} \alpha + \frac{\beta}{12\pi} - \frac{7\alpha}{180},$$

which is equivalent to (22.10) after some elementary manipulation.

ENTRY 23i. We have

$$\text{(23.1)} \qquad \frac{\phi(0)}{4\pi} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} k \coth(\pi k)(-1)^n (kx)^{4n} \phi(4n)$$

$$= \frac{\pi}{2x^2} \left\{ \frac{1}{2} \phi(2) + h \right\},$$

where the error h is nearly equal to

$$\text{(23.2)} \qquad \phi(-2) + \sum_{n=0}^{\infty} \frac{(2\pi)^{2n+1} \cos\{3(2n+1)\pi/4\} \phi(-2n-3)}{x^{2n+1}(2n+1)!},$$

if x is small. (It is not clear whether the entry reads $\phi(2)$ or $\phi(-2)$ on the right side of (23.1).)

As mentioned in the introduction, we have been unable to prove this entry. Furthermore, the interpretation intended by Ramanujan is not clear. It is surprising that a power series in x is to be approximated near $x = 0$ by a power series in $1/x$. Perhaps when h is replaced by (23.2), the difference between the left and right sides of (23.1) tends to 0 as x tends to 0. Perhaps (23.2) is an asymptotic series for h. It seems that if (23.1) is true in any sense, the hypotheses on ϕ must be quite severe and $\phi(x)$ must tend very rapidly to 0 as x tends to ∞.

It appears that (23.1) might arise from Euler's transformation of series and Newton's interpolation formula. However, such considerations have failed to establish (23.1). Possibly another transformation of series formula, for example, like those found by S. N. Aiyar [3], combined with Newton's interpolation formula will lead to a proof of (23.1). Entry 23i is somewhat reminescent of some very interesting formulae of Ramanujan on integral transforms [31, pp. 188-193], [30] for which the proofs use the aforementioned ideas.

ENTRY 23ii. We have

$$\sum_{k=1}^{\infty} \sum_{n=0}^{\infty} k^{-1} \chi(k) \operatorname{sech}(\pi k/2)(-1)^n (kx)^{4n} \phi(4n) = \frac{\pi}{8} \phi(0) - \frac{\pi}{2} h, \tag{23.3}$$

where h is very nearly equal to

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\pi/\sqrt{2})^n \phi(-n)}{x^n n!},$$

if x is small.

Comments similar to those made after Entry 23i can be made about this mysterious formula as well. However, as we shall shortly see, if we assume that the double series in (23.3) converges absolutely, then, in fact, (23.3) is indeed true with $h \equiv 0$. Of course, we are unable to make this hypothesis about the double series in (23.1).

Proof. Assume that the double series in (23.3) converges absolutely. Then inverting the order of summation and employing the Corollary of Entry 14 and Entry 15, we find that

$$\sum_{k=1}^{\infty} \sum_{n=0}^{\infty} k^{-1} \chi(k) \operatorname{sech}(\pi k/2)(-1)^n (kx)^{4n} \phi(4n)$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{4n} \phi(4n) \sum_{k=1}^{\infty} k^{4n-1} \chi(k) \operatorname{sech}(\pi k/2) = \frac{\pi}{8} \phi(0),$$

which establishes (23.3) with $h \equiv 0$.

ENTRY 24. For z complex,

$$\frac{\pi e^{-2\pi z}}{2z\{\cosh(2\pi z) - \cos(2\pi z)\}} = \frac{1}{8\pi z^3} - \frac{1}{4z^2} + \frac{\pi}{4z}$$
$$- \sum_{n=1}^{\infty} \frac{1}{z^2 + (z+n)^2} + 4z \sum_{n=1}^{\infty} \frac{n}{(e^{2\pi n} - 1)(4z^4 + n^4)}.$$

Proof. Let $f(z)$ denote the left side above. We shall expand f by partial fractions. The function f has a triple pole at $z = 0$ and simple poles at $z = \pm n(1 \pm i)/2$, where n is a positive integer. By division of power series, it is easily calculated that the principal part of f about $z = 0$ is

$$\frac{1}{8\pi z^3} - \frac{1}{4z^2} + \frac{\pi}{4z}. \tag{24.1}$$

Straightforward calculations show that

$$R\big(n(1+i)/2\big) = \frac{1}{2\,in\,(e^{2\pi n}-1)} = -\,R\big(n(1-i)/2\big)\,.$$

Replacing n by $-n$ above and manipulating slightly, we find that

$$R\big(-n(1+i)/2\big) = \frac{1}{2\,in\,(e^{2\pi n}-1)} + \frac{1}{2\,in} = -\,R\big(-n(1-i)/2\big)\,.$$

Now,

$$\text{(24.2)}\qquad \frac{1}{2\,in}\left\{\frac{1}{z+n(1+i)/2+} - \frac{1}{z+n(1-i)/2}\right\} = -\,\frac{1}{z^2+(z+n)^2}\,.$$

After much, but routine, simplification, we get

$$\text{(24.3)}\qquad \frac{1}{2\,in\,(e^{2\pi n}-1)}\left\{\frac{1}{z-n(1+i)/2} - \frac{1}{z-n(1-i)/2}\right.$$

$$\left. + \frac{1}{z+n(1+i)/2} - \frac{1}{z+n(1-i)/2}\right\} = \frac{4nz}{(e^{2\pi n}-1)(4z^4+n^4)}\,.$$

Using the principal parts in (24.1)-(24.3), we easily deduce the desired result after employing an argument like that at the end of the proof of Entry 4.

Entry 24i. For complex z we have

$$\frac{1}{2z^2} + \sum_{n=1}^{\infty} \frac{1}{z^2+n^2} = \frac{\pi}{2z} + \frac{\pi}{z\,(e^{2\pi z}-1)}\,.$$

This result is just a reformulation of (1.9).

Entry 24ii. Let z be complex. Then

$$\frac{1}{z\,(e^{\pi z}+1)} = \frac{1}{2z} - \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{z^2+(2n+1)^2}\,.$$

A proof of Entry 24ii is easily obtained by expanding the function on the left side above into partial fractions.

The next entry is complementary to Entry 24.

ENTRY 25. Let z be complex. Then

$$\frac{\pi e^{-\pi z}}{4z\{\cosh(\pi z)+\cos(\pi z)\}}=\frac{\pi}{8z}$$

$$-\sum_{n=0}^{\infty}\frac{1}{z^2+(z+2n+1)^2}-4z\sum_{n=0}^{\infty}\frac{2n+1}{(e^{(2n+1)\pi}+1)\left(4z^4+(2n+1)^4\right)}\,.$$

Proof. Let $f(z)$ denote the left side above. We expand f into partial fractions. The function f has a simple pole at $z=0$, and the principal part about 0 is easily seen to be $\pi/(8z)$. Also, f has simple poles at $z=\pm(2n+1)(1\pm i)/2$, where n is a nonnegative integer. Routine calculations give

$$R\left((2n+1)(1+i)/2\right)=\frac{i}{2(2n+1)(e^{(2n+1)\pi}+1)}$$
$$=-R\left((2n+1)(1-i)/2\right)$$

and

$$R\left(-(2n+1)(1+i)/2\right)=\frac{i}{2(2n+1)(e^{(2n+1)\pi}+1)}-\frac{i}{2(2n+1)}$$
$$=-R\left(-(2n+1)(1-i)/2\right)\,.$$

The sum of the principal parts for the four poles $\pm(2n+1)(1\pm i)/2$ is thus found to be

$$-\frac{1}{z^2+(z+2n+1)^2}-\frac{4z(2n+1)}{(e^{(2n+1)\pi}+1)(4z^4+(2n+1)^4)}\,.$$

The theorem now readily follows.

ENTRIES 25i, ii. We have

$$\sum_{k=1}^{\infty}\frac{\coth(\pi k)}{k^3}=\frac{7\pi^3}{180} \tag{25.1}$$

and

$$\sum_{k=1}^{\infty}\frac{\coth(\pi k)}{k^7}=\frac{19\pi^7}{56{,}700}\,. \tag{25.2}$$

Both (25.1) and (25.2) are special cases of the more general formula

$$\sum_{k=1}^{\infty}\frac{\coth(\pi k)}{k^{2n+1}}=2^{2n}\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\,\frac{B_{2n+2-2k}}{(2n+2-2k)!}\,. \tag{25.3}$$

where n is an odd positive integer and B_j denotes the j th Bernoulli number. Ramanujan does not state the general formula (25.3) in his Notebooks. However, it does follow quite easily from Entry 21i. (See [12, p. 155].) Formula (25.2) was communicated by Ramanujan in one of his letters to Hardy [61, p. xxvi]. Entry 25i (25.1), in fact, was long ago established by Cauchy [17, pp. 320, 361]. Cauchy does not state the general formula (25.3), but he does give a general method for evaluating the series on the left side of (25.3). Preece [54] has established (25.1) and Sandham [66] has proved (25.2). The first statement of (25.3) known to the author is by Lerch [45]. Later proofs of (25.3) have been given by Watson [73], Sandham [67], Smart [71], Sayer [68], and the author [12, p. 155], [11].

ENTRIES 25iii, iv. We have

$$\sum_{k=0}^{\infty} \frac{\tanh\{(2k+1)\pi/2\}}{(2k+1)^3} = \frac{\pi^3}{32}$$

and

$$\sum_{k=0}^{\infty} \frac{\tanh\{(2k+1)\pi/2\}}{(2k+1)^7} = \frac{7\pi^7}{23{,}040}.$$

Both entries follow from the more general formula

$$\sum_{k=0}^{\infty} \frac{\tanh\{(2k+1)\pi/2\}}{(2k+1)^{4n+3}} \tag{25.4}$$
$$= \frac{\pi^{4n+3}}{8} \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k+1}(0)\; E_{4n+1-2k}(0)}{(2k+1)!\;(4n+1-2k)!},$$

where n is a nonnegative integer and $E_j(x)$ denotes the j th Euler polynomial. Formula (25.4) cannot be found in the Notebooks. The first proof of (25.4) was given by Phillips [53]. Later proofs have been given by Nanjundiah [52], Sandham [67], Smart [71], Sayer [68], and the author [14, Corollary 4.10].

ENTRIES 25v, vi. We have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\operatorname{csch}(\pi k)}{k^3} = \frac{\pi^3}{360}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\operatorname{csch}(\pi k)}{k^7} = \frac{13\pi^7}{453{,}600}.$$

Both entries follow from the more general formula

$$(25.5) \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \operatorname{csch}(\pi k)}{k^{4n+3}}$$

$$= \frac{1}{2} (2\pi)^{4n+3} \sum_{k=0}^{2n+2} (-1)^k \frac{B_{2k}(\tfrac{1}{2})}{(2k)!} \frac{B_{4n+4-2k}(\tfrac{1}{2})}{(4n+4-2k)!},$$

where n is an integer and $B_j(x)$ denotes the jth Bernoulli polynomial. Formula (25.5) is essentially due to Cauchy [17, pp. 311, 361] who gave a somewhat less explicit formulation. Otherwise, (25.5) was first established by Mellin [50]. Later proofs have been given by Malurkar [49], Phillips [53], Nanjundiah [52], Sandham [67], Riesel [65], Sayer [68], and the author [14, Corollary 3.2]. The general formula (25.5) does not appear in the Notebooks.

Entries 25vii, viii, ix. We have

$$\sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\pi k/2)}{k} = \frac{\pi}{8},$$

$$\sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\pi k/2)}{k^5} = \frac{\pi^5}{768},$$

and

$$\sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\pi k/2)}{k^9} = \frac{23\pi^9}{1{,}720{,}320}.$$

All three entries follow from the general formula

$$(25.6) \qquad \sum_{k=1}^{\infty} \chi(k) \frac{\operatorname{sech}(\pi k/2)}{k^{4n+1}}$$

$$= \frac{1}{4} \left(\frac{\pi}{2}\right)^{4n+1} \sum_{k=0}^{2n} (-1)^k \frac{E_{2k}}{(2k)!} \frac{E_{4n-2k}}{(4n-2k)!},$$

which can be easily deduced from Entry 21ii. Here n is any integer. Entry 25vii is a simple consequence of Entry 15 and was proven by Preece [54]. Entry 25viii appeared in one of Ramanujan's letters to Hardy [61, p. xxvi]. In addition to the proofs mentioned after Entry 21ii, proofs of (25.6) have been given by Watson [73], Sandham [67], Riesel [65], and Sayer [68].

ENTRY 25x. We have

(25.7)
$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k^2 (e^{\pi k}-1)} + \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k^2 \cosh(\pi k)}$$
$$= \frac{5\pi^2}{96} - \frac{1}{2} \int_0^1 \frac{\tan^{-1} x}{x} \, dx .$$

Proof. Let

$$f(z) = \frac{1}{z^2 (e^{2\pi z}-1) \cos(\pi z)} .$$

We shall integrate f over the positively oriented square C_N whose vertical sides pass through $\pm (N + \tfrac{1}{2})\, i$, where N is a positive integer. The function f has a triple pole at $z = 0$ and simple poles at $z = \pm (2k+1)/2$, where k is a nonnegative integer, and at $z = \pm ki$, where k is a positive integer. Routine calculations yield $R(0) = 5\pi/12$,

$$R((2k+1)/2) = \frac{4(-1)^{k+1}}{(2k+1)^2 \pi (e^{(2k+1)\pi}-1)} ,$$

$$R(-(2k+1)/2) = R((2k+1)/2) + \frac{4(-1)^{k+1}}{(2k+1)^2 \pi} ,$$

and

$$R(ki) = -\frac{1}{2\pi k^2 \cosh(\pi k)} = R(-ki) .$$

Hence, applying the residue theorem and letting N tend to ∞, we find that

(25.8)
$$0 = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{C_N} f(z)\, dz = -\frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\chi(k)}{k^2 (e^{\pi k}-1)}$$
$$-\frac{4}{\pi} L(2) - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 \cosh(\pi k)} + \frac{5\pi}{12} ,$$

where $L(s)$ is defined by (21.1). A comparison of (25.8) with (25.7) indicates that it remains to show that

(25.9)
$$\int_0^1 \frac{\tan^{-1} x}{x} \, dx = L(2) .$$

Integrating termwise the Maclaurin expansion

$$\frac{\tan^{-1} x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k+1},$$

we readily deduce (25.9), and the proof is complete.

ENTRY 25xi. We have

$$\sum_{k=1}^{\infty} \frac{1}{\{k^2 + (k+1)^2\}(\cosh\{(2k+1)\pi\} - \cosh \pi)} \tag{25.10}$$
$$= \frac{1}{2\sinh \pi}\left\{\frac{1}{\pi} + \coth \pi - \frac{\pi}{2}\tanh^2(\pi/2)\right\}.$$

Entry 25xi is in error in the Notebooks, for Ramanujan has written $\sinh\{(2k+1)\pi\} - \sinh \pi$ instead of $\cosh\{(2k+1)\pi\} - \cosh \pi$ on the left side of (25.10). Ramanujan communicated (25.10), with the same error, in one of his letters to Hardy [61, p. 349]. Watson [73] established (25.10) by contour integration. Because Watson's proof contains a few errors, we briefly sketch another proof by contour integration below. The calculations in both proofs are extremely laborious.

Proof. Let

$$f(z) = \frac{\pi \sinh \pi}{z\{\cosh(\pi z) + \cosh \pi\}\{\cos(\pi z) + \cosh \pi\}},$$

which has a simple pole at $z = 0$ and poles at $z = i(2k+1) \pm 1$, if k is an integer, and at $z = 2n + 1 \pm i$, if n is an integer. These poles are simple except when $k = 0, -1$ and $n = 0, -1$ when the two sets coelesce to give double poles. Very lengthy calculations yield

$$R(0) = \frac{\pi \tanh^2(\pi/2)}{\sinh \pi},$$

$$R(i(2k+1) \pm 1)$$
$$= \frac{\pm 1}{\{i(2k+1) \pm 1\}(\cosh\{(2k+1)\pi\} - \cosh \pi)}, \quad k \neq 0, -1,$$

$$R(2n+1 \pm i)$$
$$= \frac{\pm i}{(2n+1 \pm i)(\cosh\{(2n+1)\pi\} - \cosh \pi)}, \quad n \neq 0, -1,$$

and

$$R(\pm 1 \pm i) = -\frac{\coth \pi}{2\sinh \pi} - \frac{1}{2\pi \sinh \pi}.$$

Integrate f over a rectangle with vertical and horizontal sides passing through $\pm 2N$ and $\pm 2Ni$, respectively, where N is an integer. Apply the residue theorem and let N tend to ∞ to deduce (25.10).

ENTRY 25xii. We have

$$\text{(25.11)} \qquad \sum_{k=0}^{\infty} \frac{2k+1}{\{25+(2k+1)^4/100\}(e^{(2k+1)\pi}+1)} = \frac{4689}{11{,}890} - \frac{\pi}{8}\coth^2(5\pi/2).$$

This entry again was communicated by Ramanujan in one of his letters to Hardy [61, p. 349]. The right side of (25.11), however, had the wrong sign on both terms. This error is also made in the Notebooks. Furthermore, the left side of (25.11) is replaced by only the first three terms of the series in the Notebooks, and the second term contains another misprint. It may be of interest to determine how well the first three terms on the left side of (25.11) approximate the right side. Joseph Muskat has kindly calculated that

$$\frac{1}{25.01(e^{\pi}+1)} + \frac{3}{25.81(e^{3\pi}+1)} + \frac{5}{31.25(e^{5\pi}+1)} = .001665694154\ldots,$$

while on the other hand,

$$\frac{4689}{11{,}890} - \frac{\pi}{8}\coth^2(5\pi/2) = .001665694195\ldots .$$

Watson [73] has given a proof of (25.11) by contour integration. It will be shown below that Entry 25xii is a corollary of Entry 25; hence, this is probably the method used by Ramanujan to establish (25.11).

Proof. In Entry 25 put $z = 5i$. After some simplification and rearrangement, we find that

$$\text{(25.12)} \qquad \sum_{k=0}^{\infty} \frac{2k+1}{(e^{(2k+1)\pi}+1)\{25+(2k+1)^4/100\}} - 5i\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2+10(2k+1)i-50} = -\frac{\pi\coth^2(5\pi/2)}{8}.$$

A comparison of (25.12) with (25.11) indicates that it remains to show that

$$(25.13) \qquad 5i \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 + 10(2k+1)i - 50} = \frac{4689}{11{,}890},$$

or equivalently that

$$(25.14) \qquad 50 \sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^4 + 2500} = \frac{4689}{11{,}890},$$

since (25.12) obviously implies that the imaginary part of the left side of (25.13) is zero. To show (25.14), write

$$\begin{aligned}
&50 \sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^4 + 2500} \\
&= \frac{5}{2} \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+1)^2 - 10(2k+1) + 50} - \frac{1}{(2k+1)^2 + 10(2k+1) + 50} \right\} \\
&= \frac{5}{2} \left\{ \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - 10(2k+1) + 50} \right. \\
&\qquad \left. - \sum_{k=5}^{\infty} \frac{1}{(2k+1-10)^2 + 10(2k+1-10) + 50)} \right\} \\
&= \frac{5}{2} \sum_{k=0}^{4} \frac{1}{(2k+1)^2 - 10(2k+1) + 50} = \frac{4689}{11{,}890},
\end{aligned}$$

and the proof of (25.14), and hence (25.11), is complete.

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(Reçu le 5 janvier 1979)

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