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# SOME REMARKS ON INVARIANT WHITNEY FIELDS

by Leif JACOBSEN

In this note we generalize a result of Bierstone and Milman [1] on liftings of  $\mathcal{C}^\infty$  Whitney fields to the case involving the orthogonal action of a compact Lie group  $G$ .

The method involves only completely standard notions and consists of modifications of the proof in [1]. We shall only indicate the necessary amendments and will refer to the ideas and notations of the paper [1], which should therefore be consulted all the way by the reader.

Our theorem is divined by noting that, given the action of  $G$  on  $\mathbf{R}^n$  and an invariant closed subset  $X$  of  $\mathbf{R}^n$ , one obtains a natural action on the space  $\mathcal{E}(X)$  of  $\mathcal{C}^\infty$  Whitney fields on  $X$  which leads to a very easy  $G$ -invariant version of the classical Whitney extension theorem. This action then, is the one needed in the statement of the results below (The action was implicitly used in [4] for the case  $G = S(n)$ , the permutation group). A result of Schwarz-Mather type for Whitney fields (proposition 3) is presented. We close with comments on the "remarks" of [1], as well as one or two remarks of our own.

*Notation.* The notation employed here is that of [1], which is almost identical to the one found in [6] or [9]. Thus  $X \subset \mathbf{R}^n$  is a closed set,  $J(X)$  is the space of jets  $F = (F^k)_{k \in \mathbf{N}^n}$  on  $X$ , and  $\mathcal{E}(X)$  is the subspace of Whitney fields on  $X$ . For reasons which will become apparent below, we identify  $J(X)$  with the space  $\mathcal{C}^0(X)[[z]]$  of formal power series with coefficients in the ring  $\mathcal{C}^0(X)$  of continuous functions on  $X$ , and "formal" variable  $z \in \mathbf{R}^n$ . An  $F \in J(X)$  may also be regarded as a map  $X \rightarrow \mathbf{R}[[z]]$ . The identification is given by associating  $(F^k)_{k \in \mathbf{N}^n}$  to  $\sum_{k \in \mathbf{N}^n} \frac{F^k(x)}{k!} (z-x)^k$ ,  $x \in X$ . Note that one still has an "identity" theorem and that  $\mathcal{C}^0(X)[[z]]$  is graded in  $z$ .

In addition to the concepts introduced in [1], we consider a compact Lie group  $G$ , acting orthogonally on  $\mathbf{R}^n$  (see e.g. [2] for information on group actions).  $e$  denotes the neutral element of  $G$ .

More generally, we can let  $G$  act linearly on a locally convex topological vector space  $E$ , that is via a representation  $G \rightarrow GL(E)$ , and the action is assumed to be continuous (which implies smooth for  $E = \mathbf{R}^n$ ). Often  $g \in G$  is identified with its image in  $GL(E)$ , so that  $g$  is a linear operator  $E \rightarrow E$ .

We use the same dots for all actions, so that for instance  $G \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  is written  $(g, x) \rightarrow g \cdot x$ .

If  $V$  is another locally convex vector space with a linear  $G$ -action, one gets an action on the space  $\mathcal{C}^0(E, V)$  of continuous maps  $E \rightarrow V$  by

$$(1) \quad g \cdot F = g \circ F \circ g^{-1} \quad (g \in G, F \in \mathcal{C}^0(E, V)).$$

The fixed-point set of this action is the set of  $G$ -equivariant maps, that is maps with  $F(g \cdot x) = g \cdot F(x)$ ,  $g \in G$ ,  $x \in E$ . Of course (1) also yields an action on  $\mathcal{C}^0(X, V)$  for any  $G$ -invariant subset  $X$  of  $E$  ( $G$ -invariant means  $G \cdot X = X$ ). In particular, if the action on  $V$  is trivial (which is always the case for  $V = \mathbf{R}$ ), this means that  $F(g \cdot x) = F(x)$ ,  $g \in G$ ,  $x \in E$ . One then uses the word  $G$ -invariant.

Given any  $F$  in  $\mathcal{C}^0(E, V)$ , where  $V$  is a Fréchet space, one has a corresponding equivariant map  $Av_G F$ , given by

$$(2) \quad (Av_G F)(x) = \int_G g^{-1} \cdot F(g \cdot x) d\mu(g), \quad x \in E.$$

Here  $\mu$  denotes Haar measure on  $G$ , and the vectorvalued integral is defined as in e.g. [8].

Examples: Assume that  $X$  is a closed,  $G$ -invariant subset of  $\mathbf{R}^n$ . Then there is an action on  $\mathcal{C}^0(X)$  given by

$$(3) \quad g \cdot f = f \circ g^{-1} \quad (f \in \mathcal{C}^0(X), g \in G).$$

Note that for  $X = \mathbf{R}^n$ , (3) induces an action on  $\mathcal{E}(\mathbf{R}^n)$ . Also  $Av_G f$  is smooth (continuous) whenever  $f$  is (it is here given by  $(Av_G f)(x) = \int_G f(g \cdot x) d\mu(g)$ .)

Furthermore, there is an action of  $G$  on  $J(X) = \mathcal{C}^0(X)[[z]]$  by

$$(4) \quad (g \cdot F)(z) = (g \cdot F)(g^{-1} \cdot z) \quad (g \in G, F \in J(X))$$

where  $F = (F^k)_{k \in \mathbf{N}^n}$  and  $g \cdot F = (g \cdot F^k)_{k \in \mathbf{N}^n}$ . The usual composition of power series (see [4]) is used ( $g(0) = 0$ ). Let  $\mathcal{E}(\mathbf{R}^n)^G$  and  $\mathcal{C}^0(X)[[z]]^G$  denote the fixed-point subspaces of  $G$ -invariant functions and jets and put  $\mathcal{E}(X)^G = \mathcal{C}^0(X)[[z]]^G \cap \mathcal{E}(X)$ . When  $z$  is left out, the fact that  $F$  is an invariant jet means that certain linear relations between the  $F^k \circ g$

hold, for each fixed  $|k|$ . The coefficients are powers of the entries  $g_{ij}$  of  $g \in GL(n, \mathbf{R})$ . In particular,  $F^0$  is  $G$ -invariant.

We have the map  $T_X = J : \mathcal{E}(\mathbf{R}^n) \rightarrow J(X) = \mathcal{C}^0(X) [[z]]$ . For convenience, we put  $J_x(f) = J(f)(x)$ ,  $f \in \mathcal{E}(\mathbf{R}^n)$ ,  $x \in X$ . Viewing  $F$  in  $J(X)$  as a map  $X \rightarrow \mathbf{R} [[z]]$  and using (2), (3), (4) and the fact that  $J_x(f \circ g) = J_{g(x)}(f) \circ g_x$  (here  $g_x(z) = g \cdot x + g \cdot (z - x)$ ), one computes that  $J \circ AV_G = AV_G \circ J$ . Note that in the space  $V = \mathcal{C}^0(X) [[z]]$ , the definition of  $\mu$  means that  $\int \sum_k F^k(g, x) z^k d\mu(g) = \sum_k [\int F^k(g, x) d\mu(g)] z^k$  ( $F^k \in \mathcal{C}^0(G \times X)$ ). Also note that  $J_x(f)$  is graded by the  $J_x^m(f) = \sum_{|k| \leq m} \frac{D^k f(x)}{k!} (z - x)^k$ ,  $m \in \mathbf{N}$ . Let  $T_X^m(f) = (D^k f | X)_{|k| \leq m}$ . The reason for introducing the actions (3) and (4) becomes clear in the following two propositions.  $G$  is acting orthogonally on  $\mathbf{R}^n$ .

PROPOSITION 1. Let  $X \subset \mathbf{R}^n$  be  $G$ -invariant. Then

$$T_X : \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(X)$$

is  $G$ -equivariant. Consequently  $\mathcal{E}(\mathbf{R}^n)^G$  is mapped into  $\mathcal{E}(X)^G$ .

*Proof.* The action (4) grades  $\mathcal{C}^0(X) [[z]]$  as well as the subring  $\mathcal{E}(X)$ , so we may prove the proposition by induction. Assume that  $J_x^m(g \cdot f) = g \cdot J_x^m(f)$  is true for  $g \in G$ ,  $x \in X$ ,  $f \in \mathcal{E}(\mathbf{R}^n)$ . Any  $k \in \mathbf{N}^n$  with  $|k| = m + 1$  is of the form  $k = k' + (i)$ ,  $|k'| = m$  (see [1] p. 135 for (i)). For all  $x \in X$ ,  $g \in G$ , one has

$$(a) \quad \sum_{i=1}^n D_i(f \circ g)(x) (z - x)_i = \sum_{i=1}^n (D_i f) \circ g(x) [g \cdot (z - x)]_i,$$

noticing that  $[g \cdot (z - x)]_i = g_{|i}^{-1} \cdot (z - x)$  ( $g$  is orthogonal).

Here  $D_i = D^{(i)}$  and  $g_{|i}$  is the  $i$ 'th column of  $g \in G$ . Now  $J_x^{m+1}(f) - J_x^m(f) = \sum_{i=1}^n \sum_{|k'|=m} \frac{(D_i D^{k'} f)(x)}{(k' + (i))!} (z - x)^{k'} (z - x)_i$ , hence induction combined with (a) completes the proof.

PROPOSITION 2. Let  $X \subset \mathbf{R}^n$  be  $G$ -invariant.  $T_X : \mathcal{E}(\mathbf{R}^n)^G \rightarrow \mathcal{E}(X)^G$  is surjective.  $T_X^m : \mathcal{E}(\mathbf{R}^n)^G \rightarrow \mathcal{E}^m(X)^G$  is split-surjective for all  $m \in \mathbf{N}$ .

Here  $\mathcal{E}^m(X)$  denotes the subspace of Whitney fields  $(F^k)_{k \in \mathbf{N}^n}$  with  $F^k = 0$  for  $|k| > m$ . See [5] p. 146 for the definition of split-surjective.



*Proof.* By the Whitney extension theorem there is a function  $\tilde{f} \in \mathcal{E}(\mathbf{R}^n)$  with  $J(\tilde{f}) = F$  for a given  $F \in \mathcal{E}(X)^G$ .

Put  $f = Av_G \tilde{f} = \int_G \tilde{f}(g \cdot x) d\mu(g)$ . Then  $f \in \mathcal{E}(\mathbf{R})^G$  and  $J(f) = Av_G J(\tilde{f}) = Av_G F = F$  by the last remarks in the section on notation.

For  $m < \infty$  we may choose  $\tilde{f} = \Phi_m(F)$ ,  $\Phi_m$  being continuous and linear, and so  $Av_G \circ \Phi_m$  splits  $T_X^m$ .

This is a natural invariant version of the Whitney extension theorem. Now we are prepared to generalize the theorem of [1].

**THEOREM.** Let  $X$  be a  $G$ -invariant closed subset of  $\mathbf{R}^n$ , and  $E$  a Hausdorff topological vectorspace, topologized by a family of seminorms  $\|\cdot\|_{\lambda \in \Lambda}$ . Assume that  $G$  acts linearly and continuously on  $E$ . Let  $H : E \rightarrow \mathcal{E}(X)$  be an equivariant continuous linear map. Suppose that for each  $a \in X$ , there is a continuous linear map  $H_a : E \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that

$$(a') \quad T_X H_a(\xi) = H(\xi)(a) \quad \text{for all } \xi \in E$$

(b') For each  $m \in \mathbf{N}$  and  $L \subset \mathbf{R}^n$  compact, there exists  $\lambda = \lambda(m, L) \in \Lambda$  and a constant  $c = c(m, L)$  such that for all  $\xi \in E$

$$(2') \quad \|H_a(\xi)\|_m^L \leq c(m, L) \|\xi\|_{\lambda(m, L)}.$$

Then there exists an equivariant continuous linear map  $\tilde{H} : E \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $\tilde{H}(\xi)|_X = H(\xi)$ ,  $\xi \in E$  (that is,  $T_X \tilde{H} = H$ ).

Here the assumption is that  $G$  acts on  $\mathcal{E}(\mathbf{R}^n)$  and  $\mathcal{E}(X)$  by (3) and (4). a') expresses an identity in  $\mathbf{R}[[z]]$  (this is also the meaning of a) in [1]).

Now let  $F : \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(X)$  be a continuous linear map and denote by  $\text{supp } F$  its support as defined in the natural way ([1] p. 132). Let  $G$  act linearly and continuously on  $\mathcal{E}(\mathbf{R}^k)$ . Assume that  $F$  is  $G$ -equivariant, and note that then  $\text{supp } F$  is invariant. By the proof of the corollary 1 in [1], we have

**COROLLARY 1.** If  $F$  has compact support, then there is an equivariant continuous linear map  $\tilde{F} : \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)$ , such that the following diagram commutes

$$\begin{array}{ccc} & \tilde{F} & \mathcal{E}(\mathbf{R}^n) \\ & \nearrow & \downarrow T_X \\ \mathcal{E}(\mathbf{R}^k) & \xrightarrow{F} & \mathcal{E}(X) \end{array}$$

In particular, if  $G$  acts trivially on  $\mathcal{E}(\mathbf{R}^k)$ , we have a commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\sim F} & \mathcal{E}(\mathbf{R}^n)^G \\ & & \downarrow T_X \\ \mathcal{E}(\mathbf{R}^k) & \xrightarrow{F} & \mathcal{E}(X) \end{array}$$

by proposition 1.

Now consider the situation described in [5]: Let  $\sigma : \mathbf{R}^n \rightarrow \mathbf{R}^k$  be the (proper) Hilbert polynomial map. This is the map given by a set  $(\sigma_1, \dots, \sigma_k)$  of (minimal) generators for the algebra of  $G$ -invariant polynomials  $\mathbf{R}^n \rightarrow \mathbf{R}$  (see also [7], p. 6). The Schwarz-Mather theorem states that the map  $\sigma^* : \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)^G$ ,  $H \mapsto H \circ \sigma$  is split-surjective. Correspondingly, we have for  $X \subset \mathbf{R}^n$   $G$ -invariant the

PROPOSITION 3. The mapping  $\sigma^* : \mathcal{E}(\sigma(X)) \rightarrow \mathcal{E}(X)^G$  is split-surjective.

*Proof.* The composition  $H \circ \sigma$ ,  $H \in \mathcal{E}(\sigma(X))$  is as in [4]. As in [5] lemma 3, we let  $\mathcal{H}(X)_d^G$  denote the space of homogeneous invariant fields of degree  $d$  (this makes sense, namely for each  $x \in X$ ) and  $\mathcal{W}(\sigma(X))_d$  the space of weighted homogeneous Whitney fields of degree  $d$  on  $\sigma(X)$ . Now put  $\bar{\eta}_m = T_{\sigma(X)}^m \circ \eta \circ \Phi_m$ ,  $m \in \mathbf{N}$ , where  $\eta$  is chosen to split

$$\sigma^* : \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)^G \quad \text{and} \quad \Phi_m$$

splits  $T_X^m$  according to proposition 2.

$$\begin{array}{ccc} \mathcal{E}(\mathbf{R}^n)^G & \xrightleftharpoons[\Phi_m]{T_X^m} & \mathcal{E}^m(X)^G \\ \sigma^* \uparrow \eta \downarrow & & \uparrow \sigma_m^* \\ \mathcal{E}(\mathbf{R}^k) & \xrightarrow{T_{\sigma(X)}} & \mathcal{E}(\sigma(X)) \end{array}$$

Evidently  $\bar{\eta}_m$  splits  $\sigma_m^*$  by the commutative diagram. Then one derives that  $\sigma_d^* : \mathcal{W}(\sigma(X))_d \rightarrow \mathcal{H}(X)_d^G$  is split-surjective, whence

$$\sigma^* = \prod_d \sigma_d^* : \prod_d \mathcal{W}(\sigma(X))_d \rightarrow \prod_d \mathcal{H}(X)_d^G$$

is split-surjective (the rings are graded via  $\sigma$ ). Note that the resulting map  $\bar{\eta}$  is continuous using the topology on  $\mathcal{E}(X)$  given in [1].

It was tacitly used that  $H \circ \sigma$  is Whitney if  $H$  is (noted in [4]).

*Proof of the Theorem.* We trace the proof in [1], with the necessary modifications.

It suffices to prove the theorem for  $X = K$  compact, because we have  $G$ -invariant continuous partitions of unity on  $X$ , using  $A\nu_G$ .

Take the Whitney partition of unity  $\{\Phi_i \mid i \in I\}$  on  $\mathbf{R}^n \setminus K$  from [1] and put  $\tilde{\Phi}_i = A\nu_G \Phi_i$ ,  $i \in I$ . Then the family  $\{\tilde{\Phi}_i \mid i \in I\}$  of functions in  $\mathcal{E}(\mathbf{R}^n \setminus K)^G$  has the properties

- i')  $\{\text{supp } \tilde{\Phi}_i \mid i \in I\}$  is a locally finite family and  $N(x) \leq N(n, G)$ , a number depending on  $G$  and  $n$ , but not on  $x$ .
- ii')  $\tilde{\Phi}_i \geq 0$  for all  $i \in I$ .  $\sum_{i \in I} \tilde{\Phi}_i(x) = 1$  for all  $x \in \mathbf{R}^n \setminus K$ .
- iv') There exists a constant  $\tilde{c}_k$ , depending only on  $k, n$  and  $G$ , such that for all  $x \in \mathbf{R}^n \setminus K$

$$|D^k \tilde{\Phi}_i(x)| < \tilde{c}_k (1 + d(x, K))^{-|k|}$$

Here ii') is clear by the properties of Haar measure.

For all  $g \in G$ , induction and the chain rule shows that

$$D^k(\Phi_i \circ g)(x) = L_g^k(\Phi_i) \circ g(x),$$

where  $L_g^k$  is a linear partial differential operator of order  $|k|$  with coefficients depending polynomially on  $g$ . If  $C_G^k$  is the supremum of all these coefficients over  $G$ , and  $N_k$  their number, then the definition  $\tilde{\Phi}_i(x) = \int_G \Phi_i \circ g(x) d\mu(g)$  and the inequality valid for  $\Phi_i$  shows that

$$|D^k \tilde{\Phi}_i(x)| \leq N_k C_G^k C_k (1 + d(x, K))^{-|k|},$$

because we have  $d(g \cdot x, K) \geq d(x, K)$  for all  $g \in G$  ( $g$  is orthogonal). This proves iv').

i') is proved by induction on  $\dim G$ . It is evident for  $\dim G = 0$  ( $G$  being then discrete, hence finite), because  $\text{supp } \tilde{\Phi}_i \subset G$ .  $\text{supp } \Phi_i$ .

Suppose the claim is true for all  $p < \dim G$ . Taking any  $x \in \mathbf{R}^n$ , the slice theorem (see [2], p. 308) enables us to look at a  $G$ -invariant neighborhood of  $x$  as being of the form  $Gx_H V$ ,  $H = G_x$  the isotropy group at  $x$ ,  $V = V_x$  the normal space to the orbit  $G(x)$  with orthogonal  $H$ -action.

There is a trivial isomorphism  $\mathcal{E}(Gx_H V)^G \simeq \mathcal{E}(V)^H$  (see e.g. [7], p. 51), so that each restriction  $\tilde{\Phi}_i|_{Gx_H V}$  may be looked upon as an  $H$ -invariant function on  $V$ . Assuming that  $G(x)$  is not discrete, hence  $\dim H < \dim G$ , the induction can be carried out.

[The inequality iii) has no counterpart; one may show  $6 d(\text{supp } \tilde{\Phi}_i, K) + \delta \geq \text{diam}(\text{supp } \tilde{\Phi}_i)$ ,  $\delta = \text{diam } G(a_i)$  for an  $a_i \in K$  realizing the distance to  $\text{supp } \Phi_i$ ].

Now we define  $f = \tilde{H}(\xi) \in \mathcal{E}(\mathbf{R}^n)$  in a manner similar to the one in [1], by

$$f(x) = F^0(x), \quad x \in K$$

$$f(x) = \sum_{i \in I} \tilde{\Phi}_i(x) (Av_G H_{a_i})(\xi)(x), \quad x \notin K$$

The  $a_i$  are chosen as in [1] (note that  $d(K, \text{supp } \Phi_i) = d(K, \text{supp } \tilde{\Phi}_i)$ ).  $Av_G H_{a_i}$  and  $\tilde{\Phi}_i$  being respectively equivariant and invariant,  $\tilde{H}$  becomes equivariant according to (3).

We now show that the continuous linear maps  $Av_G H_a$  are pointwise lifts, that is, still fulfil a') and b') ( $c$  now depending on  $G$  too)

For  $a \in K$ ,  $\xi \in E$  we have

$$\begin{aligned} T_K(Av_G H_a)(\xi)(a) &= T_K \int_G g^{-1} \cdot H_a(g \cdot \xi) dg(a) \\ &= \int_G g^{-1} \cdot T_K(H_a(g \cdot \xi))(a) dg = \int_G g^{-1} \cdot H(g \cdot \xi)(a) dg \\ &= \int_G H(\xi)(a) dg = H(\xi)(a), \end{aligned}$$

using a') and the fact that  $H$  and  $T_K$  are equivariant (prop. 1).  $dg = d\mu(g)$  here.

Now, to tackle b'), one first observes that, as is well known, one may (without loss) assume the family  $\|\cdot\|_{\lambda \in \Lambda}$  defining the locally convex topology on  $E$ , to be upward filtering. (2') can still be assumed to hold, and the continuity of an operator  $p: E \rightarrow E$  means that to each  $\|\cdot\|_{\lambda}$  there is a constant  $c$  and a  $\mu \in \Lambda$  such that

$$\|p(\xi)\|_{\lambda} \leq c \|\xi\|_{\mu} \quad \text{for all } \xi \in E.$$

Take  $\lambda \in \Lambda$ . Then for all  $\xi \in E$  one has an inequality

$$(x) \quad \|g \cdot \xi\|_{\lambda} \leq c' \|\xi\|_{\lambda'}$$

for some  $\lambda' \in \Lambda$  and a constant  $c' = c'(G, \lambda)$ , but *independent* of  $g \in G$ .

In fact, given  $\|\cdot\|_{\lambda}$  and  $\varepsilon = 1$ , there is a neighborhood  $U$  of  $e$  in  $G$  and  $\lambda'' \in \Lambda$ ,  $\delta > 0$  such that  $g \in U$ ,  $\|\xi\|_{\lambda''} \leq \delta$  implies  $\|g \cdot \xi\|_{\lambda} \leq 1$ , by the continuity of the action. Let  $G$  be covered by finitely many left-translates  $g_j U$ ,  $j \in J$ . To each  $g_j$ , viewed as an element of  $GL(E)$ , there is a constant  $c_j$  and  $\lambda_j \in \Lambda$  such that  $\|g_j \cdot \xi\|_{\lambda''} \leq c_j \|\xi\|_{\lambda_j}$  for all  $\xi \in E$ . Put  $c = \sup_{j \in J} c_j$  and choose  $\lambda' \in \Lambda$  such that  $\|\cdot\|_{\lambda'} \geq \|\cdot\|_{\lambda_j}$ ,  $j \in J$ .

Now, given  $g \in G$ ,  $\xi \in E$ , we put  $k = \|\xi\|_{\lambda'}$ . It may be assumed that  $k > 0$ . Let  $h = ck/\delta$ , then  $\|h^{-1}g_j\xi\|_{\lambda''} \leq \delta$  for all  $j \in J$ . Furthermore,  $gg_j^{-1} \in U$  for some  $j \in J$ . Hence  $\|g \cdot \xi\|_{\lambda} = h \|(gg_j^{-1})(h^{-1}g_j\xi)\|_{\lambda} \leq h = (c\delta^{-1}) \|\xi\|_{\lambda'}$ .

Next, using (x), (2'), the estimates of (iv') and the Haar integral, one gets

$$\begin{aligned} |Av_G H_a(\xi)|_m^L &= \sup_{\substack{x \in L \\ |k| \leq m}} |D^k \int_G g^{-1} \cdot H_a(g \cdot \xi)(x) dg| \\ &= \sup_{\substack{x \in L \\ |k| \leq m}} |\int_G D^k (H_a(g \cdot \xi) \circ g)(x) dg| \\ &\leq \int_G \sup_{\substack{x \in L \\ |k| \leq m}} |[L_g^k (H_a(g \cdot \xi))] \circ g(x)| dg \\ &\leq \int_G N_m C_G^m |H_a(g \cdot \xi)|_{m, L}^{G \cdot L} dg \\ &\leq \int_G N_m C_G^m C(m, G \cdot L) \|g \cdot \xi\|_{\lambda(m, G \cdot L)} dg \\ &\leq C'(m, L, G) \|\xi\|_{\lambda'(m, L, G)} \end{aligned}$$

for some  $C' = C'(m, L, G)$  ( $N_m = \sup_{|k| \leq m} N_k$  etc.).

This proves b') for  $Av_G H_a$ .

Now the rest of the proof can be carried out as in [1], replacing  $G_a$  with  $Av_G H_a$ : the evaluations (4)-(7) are valid without change and so only the *claim* of [1] with  $Av_G H_a$  substituted for  $G_a$  must be proved. This too goes through as in [1], using a'), b'), (4)-(7) from [1] and i'), ii'), iv').

At two points (estimation of  $|S_0(x)|$  and  $|S_1(x)|$ ) the inequality iii) is needed, and as this is a purely local matter, necessary only to obtain the inequalities  $|x - a_i| \leq 3|x - a|$ ,  $|a - a_i| \leq 4|x - a|$  ( $x \in \mathbf{R}^n \setminus K$ ,  $a \in K$ ) the estimate iii), valid for the original  $\Phi_i$ , can be used again, because we could choose  $a_i$  in  $\text{supp } \Phi_i$ .

*Remark.* If  $E$  is a Fréchet space it is only necessary to assume that the action  $G \times E \rightarrow E$  is separately continuous. Indeed, the boundedness of orbits  $G \cdot \xi$  implies via the Banach-Steinhaus theorem that  $\{\xi \mapsto g \cdot \xi\}_{g \in G}$  is an equicontinuous set of operators, hence for a  $\delta > 0$  chosen as above (no  $U$  needed) one gets  $\|g \cdot \xi\|_{\lambda} \leq \delta^{-1} \|\xi\|_{\lambda}$ , ( $g \in G$ ,  $\xi \in E$ ) instead of (x).

In the remark 5 of [1] the possibility of obtaining the pointwise lifts  $H_a$  via finite map-germs  $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}^p$  is discussed. We point out that there are  $G$ -equivariant extensions of these theories as developed in [7]; in particular the equivariant version of the preparation theorem is true [7], p. 64-72. Thus if the  $X$ ,  $X'$  and  $\varphi$  mentioned in that remark are invariant under the

orthogonal action of  $G$ , the  $W'$  and  $W$  may already be chosen equivariant.

Similarly, it appears that there is a  $G$ -invariant version of the Stein extension theorem mentioned in remark 4 of [1]. This results from an invariant version of corollary 2 of [1], combined with the invariant Seeley extension theorem [7], p. 108. The  $X$  in the remark 4 is invariant if  $\varphi$  is ( $G$  acts on  $\mathbf{R}^{n+1}$  by  $g \cdot (x, y) = (g \cdot x, y)$  for  $g \in G$ ).

An alternative approach would be via proposition 2 and the techniques of [4], which are somewhat similar.

We also wish to point out that there is a  $G$ -invariant version of the Whitney spectral theorem (see [6], ch. V or [9], ch. V):

Let  $\Omega \subset \mathbf{R}^n$  be open and invariant under an orthogonal action of  $G$ ,  $G$  compact Lie. Let  $I \subset \mathcal{E}(\Omega)^G$  (using the action (3)) be an ideal. Then  $f \in \mathcal{E}(\Omega)^G$  belongs to  $\bar{I}$  if and only if for each  $a \in \Omega$  there is a  $g_a = g \in I$  such that  $J_a(g) = J_a(f)$ .

This goes via a fundamental lemma [6], p. 91, for the case  $\Omega = a$  cube  $L$ . With the notations of that lemma, if  $L$  is replaced by  $G \cdot L$ ,  $K$  by  $G \cdot K$  and  $T_b^m$  by  $Av_G T_b^m$ , then  $F \in \hat{I}$  may be assumed invariant on  $G \cdot L$ , whence  $\|\tilde{\Phi} F - f\|_{G \cdot L}^m < \varepsilon$ , ( $\tilde{\Phi} = Av_G \Phi$ ) can be achieved. Then one proceeds. In the more general situation considered in [9], one needs [7], lemma 1.4.1 (p. 106).

The action (4) is adapted to the operators  $D^k$ . One might consider the simpler action on  $J(X)$  ( $X \subset \mathbf{R}^n$   $G$ -invariant), given by  $g \cdot F = (F^k \circ g^{-1})_{k \in \mathbf{N}^n}$ , for  $F = (F^k)_{k \in \mathbf{N}^n}$ ,  $g \in G$ . The corresponding problem of finding  $f$  with  $J(f) = F$ , given  $F \in \mathcal{E}(X)^G$ , is now wholly different as simple examples show (e.g.  $G = \mathbf{Z}_2$  acting by reflexion in  $0 \in \mathbf{R}$ ). If  $f$  exists at all, it must have strong singularities on  $K$ . As may be gleaned from [3], there are topological restrictions on  $K$ , depending on  $G$ . It would perhaps be feasible to obtain some answers if new operators are used instead of the  $D^k$ .

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