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SOME REMARKS ON INVARIANT WHITNEY FIELDS

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In this note we generalize a result of Bierstone and Milman [1] on liftings of $\mathscr{C}^\infty$ Whitney fields to the case involving the orthogonal action of a compact Lie group G.

The method involves only completely standard notions and consists of modifications of the proof in [1]. We shall only indicate the necessary amendments and will refer to the ideas and notations of the paper [1], which should therefore be consulted all the way by the reader.

Our theorem is divined by noting that, given the action of G on $\mathbf{R}^n$ and an invariant closed subset X of $\mathbf{R}^n$, one obtains a natural action on the space $\mathscr{E}(X)$ of $\mathscr{C}^\infty$ Whitney fields on X which leads to a very easy G-invariant version of the classical Whitney extension theorem. This action then, is the one needed in the statement of the results below (The action was implicitly used in [4] for the case $G = S(n)$, the permutation group). A result of Schwarz-Mather type for Whitney fields (proposition 3) is presented. We close with comments on the "remarks" of [1], as well as one or two remarks of our own.

Notation. The notation employed here is that of [1], which is almost identical to the one found in [6] or [9]. Thus $X \subset \mathbf{R}^n$ is a closed set, $J(X)$ is the space of jets $F = (F^k)_{k \in \mathbf{N}^n}$ on X, and $\mathscr{E}(X)$ is the subspace of Whitney fields on X. For reasons which will become apparent below, we identify $J(X)$ with the space $\mathscr{C}^0(X)[[z]]$ of formal power series with coefficients in the ring $\mathscr{C}^0(X)$ of continuous functions on X, and "formal" variable $z \in \mathbf{R}^n$. An $F \in J(X)$ may also be regarded as a map $X \to \mathbf{R}[[z]]$. The identification is given by associating $(F^k)_{k \in \mathbf{N}^n}$ to $\sum\limits_{k \in \mathbf{N}^n} \dfrac{F^k(x)}{k!}(z-x)^k$, $x \in X$. Note that one still has an "identity" theorem and that $\mathscr{C}^0(X)[[z]]$ is graded in z.

In addition to the concepts introduced in [1], we consider a compact Lie group G, acting orthogonally on $\mathbf{R}^n$ (see e.g. [2] for information on group actions). e denotes the neutral element of G.

More generally, we can let G act linearly on a locally convex topological vector space E, that is via a representation $G \to GL(E)$, and the action is assumed to be continuous (which implies smooth for $E = \mathbf{R}^n$). Often $g \in G$ is identified with its image in $GL(E)$, so that g is a linear operator $E \to E$.

We use the same dots for all actions, so that for instance $G \times \mathbf{R}^n \to \mathbf{R}^n$ is written $(g, x) \to g \,.\, x$.

If V is another locally convex vector space with a linear G-action, one gets an action on the space $\mathscr{C}^0(E, V)$ of continuous maps $E \to V$ by

(1) $$g \,.\, F = g \circ F \circ g^{-1} \quad (g \in G,\ F \in \mathscr{C}^0(E, V)).$$

The fixed-point set of this action is the set of *G-equivariant* maps, that is maps with $F(g \,.\, x) = g \,.\, F(x)$, $g \in G$, $x \in E$. Of course (1) also yields an action on $\mathscr{C}^0(X, V)$ for any G-invariant subset X of E (G- invariant means $G \,.\, X = X$). In particular, if the action on V is trivial (which is always the case for $V = \mathbf{R}$), this means that $F(g \,.\, x) = F(x)$, $g \in G$, $x \in E$. One then uses the word *G-invariant.*

Given any F in $\mathscr{C}^0(E, V)$, where V is a Fréchet space, one has a corresponding equivariant map $Av_G\, F$, given by

(2) $$(Av_G\, F)(x) = \int_G g^{-1} \,.\, F(g \,.\, x)\, d\mu(g), \quad x \in E.$$

Here μ denotes Haar measure on G, and the vectorvalued integral is defined as in e.g. [8].

Examples: Assume that X is a closed, G-invariant subset of $\mathbf{R}^n$. Then there is an action on $\mathscr{C}^0(X)$ given by

(3) $$g \,.\, f = f \circ g^{-1} \quad (f \in \mathscr{C}^0(X),\ g \in G).$$

Note that for $X = \mathbf{R}^n$, (3) induces an action on $\mathscr{E}(\mathbf{R}^n)$. Also $Av_G f$ is smooth (continuous) whenever f is (it is here given by $(Av_G f)(x) = \int_G f(g \,.\, x)\, d\mu(g)$.)

Furthermore, there is an action of G on $J(X) = \mathscr{C}^0(X)[[z]]$ by

(4) $$(g \,.\, F)(z) = (g \,.\, F)(g^{-1} \,.\, z) \quad (g \in G,\ F \in J(X))$$

where $F = (F^k)_{k \in \mathbf{N}^n}$ and $g \,.\, F = (g \,.\, F^k)_{k \in \mathbf{N}^n}$. The usual composition of power series (see [4]) is used $(g(0) = 0)$. Let $\mathscr{E}(\mathbf{R}^n)^G$ and $\mathscr{C}^0(X)[[z]]^G$ denote the fixed-point subspaces of G-invariant functions and jets and put $\mathscr{E}(X)^G = \mathscr{C}^0(X)[[z]]^G \cap \mathscr{E}(X)$. When z is left out, the fact that F is an invariant jet means that certain linear relations between the $F^k \circ g$

hold, for each fixed $|k|$. The coefficients are powers of the entries g_{ij} of $g \in GL(n, \mathbf{R})$. In particular, F^0 is G-invariant.

We have the map $T_X = J : \mathscr{E}(\mathbf{R}^n) \to J(X) = \mathscr{C}^0(X)[[z]]$. For convenience, we put $J_x(f) = J(f)(x), f \in \mathscr{E}(\mathbf{R}^n), x \in X$. Viewing F in $J(X)$ as a map $X \to \mathbf{R}[[z]]$ and using (2), (3), (4) and the fact that $J_x(f \circ g) = J_{g(x)}(f) \circ g_x$ (here $g_x(z) = g \cdot x + g \cdot (z-x)$), one computes that $J \circ AV_G = AV_G \circ J$. Note that in the space $V = \mathscr{C}^0(X)[[z]]$, the definition of μ means that $\int \sum_k F^k(g, x) z^k d\mu(g) = \sum_k [\int F^k(g, x) d\mu(g)] z^k$ $(F^k \in \mathscr{C}^0(G \times X).)$. Also note that $J_x(f)$ is graded by the $J_x^m(f) = \sum_{|k| \leq m} \frac{D^k f(x)}{k!}(z-x)^k$, $m \in \mathbf{N}$. Let $T_X^m(f) = (D^k f \mid X)_{|k| \leq m}$. The reason for introducing the actions (3) and (4) becomes clear in the following two propositions. G is acting orthogonally on $\mathbf{R}^n$.

PROPOSITION 1. Let $X \subset \mathbf{R}^n$ be G-invariant. Then

$$T_X : \mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(X)$$

is G-equivariant. Consequently $\mathscr{E}(\mathbf{R}^n)^G$ is mapped into $\mathscr{E}(X)^G$.

Proof. The action (4) grades $\mathscr{C}^0(X)[[z]]$ as well as the subring $\mathscr{E}(X)$, so we may prove the proposition by induction. Assume that $J_x^m(g \cdot f) = g \cdot J_x^m(f)$ is true for $g \in G$, $x \in X$, $f \in \mathscr{E}(\mathbf{R}^n)$. Any $k \in \mathbf{N}^n$ with $|k| = m + 1$ is of the form $k = k' + (i)$, $|k'| = m$ (see [1] p. 135 for (i)). For all $x \in X$, $g \in G$, one has

$$\text{(a)} \quad \sum_{i=1}^{n} D_i(f \circ g)(x)(z-x)_i = \sum_{i=1}^{n} (D_i f) \circ g(x)[g \cdot (z-x)]_i,$$

noticing that $[g \cdot (z-x)]_i = g^{-1}_{|i} \cdot (z-x)$ (g is orthogonal).

Here $D_i = D^{(i)}$ and $g|_i$ is the i'th column of $g \in G$. Now $J_x^{m+1}(f) - J_x^m(f) = \sum_{i=1}^{n} \sum_{|k'|=m} \frac{(D_i D^{k'} f(x)}{(k'+(i))!}(z-x)^{k'}(z-x)_i$, hence induction combined with (a) completes the proof.

PROPOSITION 2. Let $X \subset \mathbf{R}^n$ be G-invariant. $T_X : \mathscr{E}(\mathbf{R}^n)^G \to \mathscr{E}(X)^G$ is surjective. $T_X^m : \mathscr{E}(\mathbf{R}^n)^G \to \mathscr{E}^m(X)^G$ is split-surjective for all $m \in \mathbf{N}$.

Here $\mathscr{E}^m(X)$ denotes the subspace of Whitney fields $(F^k)_{k \in \mathbf{N}^n}$ with $F^k = 0$ for $|k| > m$. See [5] p. 146 for the definition of split-surjective.

Proof. By the Whitney extension theorem there is a function $\tilde{f} \in \mathscr{E}(\mathbf{R}^n)$ with $J(\tilde{f}) = F$ for a given $F \in \mathscr{E}(X)^G$.

Put $f = Av_G \tilde{f} = \int_G \tilde{f}(g \,.\, x)\, d\mu(g)$. Then $f \in \mathscr{E}(\mathbf{R})^G$ and $J(f) = Av_G J(\tilde{f}) = Av_G F = F$ by the last remarks in the section on notation. For $m < \infty$ we may choose $\tilde{f} = \Phi_m(F)$, Φ_m being continuous and linear, and so $Av_G \circ \Phi_m$ splits T_X^m.

This is a natural invariant version of the Whitney extension theorem. Now we are prepared to generalize the theorem of [1].

THEOREM. Let X be a G-invariant closed subset of $\mathbf{R}^n$, and E a Hausdorff topological vectorspace, topologized by a family of seminorms $\|\cdot\|_{\lambda \in \Lambda}$. Assume that G acts linearly and continuously on E. Let $H : E \to \mathscr{E}(X)$ be an equivariant continuous linear map. Suppose that for each $a \in X$, there is a continuous linear map $H_a : E \to \mathscr{E}(\mathbf{R}^n)$ such that

(a′) $T_X H_a(\xi) = H(\xi)(a)$ for all $\xi \in E$

(b′) For each $m \in \mathbf{N}$ and $L \subset \mathbf{R}^n$ compact, there exists $\lambda = \lambda(m, L) \in \Lambda$ and a constant $c = c(m, L)$ such that for all $\xi \in E$

(2′) $|H_a(\xi)|_m^L \leqslant c(m, L) \|\xi\|_{\lambda(m,L)}$.

Then there exists an equivariant continuous linear map $\tilde{H} : E \to \mathscr{E}(\mathbf{R}^n)$ such that $\tilde{H}(\xi) \mid X = H(\xi)$, $\xi \in E$ (that is, $T_X \tilde{H} = H$).

Here the assumption is that G acts on $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{E}(X)$ by (3) and (4). a′) expresses an identity in $\mathbf{R}[[z]]$ (this is also the meaning of a) in [1]).

Now let $F : \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(X)$ be a continuous linear map and denote by supp F its support as defined in the natural way ([1] p. 132). Let G act linearly and continuously on $\mathscr{E}(\mathbf{R}^k)$. Assume that F is G-equivariant, and note that then supp F is invariant. By the proof of the corollary 1 in [1], we have

COROLLARY 1. If F has compact support, then there is an equivariant continuous linear map $\tilde{F} : \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)$, such that the following diagram commutes

$$\begin{tikzcd} & \mathscr{E}(\mathbf{R}^n) \arrow[d, "T_X"] \\ \mathscr{E}(\mathbf{R}^k) \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"] & \mathscr{E}(X) \end{tikzcd}$$

In particular, if G acts trivially on $\mathscr{E}(\mathbf{R}^k)$, we have a commutative diagram

$$\begin{tikzcd} & \mathscr{E}(\mathbf{R}^n)^G \arrow[d, "T_X"] \\ \mathscr{E}(\mathbf{R}^k) \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"'] & \mathscr{E}(X) \end{tikzcd}$$

by proposition 1.

Now consider the situation described in [5]: Let $\sigma : \mathbf{R}^n \to \mathbf{R}^k$ be the (proper) Hilbert polynomial map. This is the map given by a set $(\sigma_1, \ldots, \sigma_k)$ of (minimal) generators for the algebra of G-invariant polynomials $\mathbf{R}^n \to \mathbf{R}$ (see also [7], p. 6). The Schwarz-Mather theorem states that the map $\sigma^* : \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)^G$, $H \to H \circ \sigma$ is split-surjective. Correspondingly, we have for $X \subset \mathbf{R}^n$ G-invariant the

PROPOSITION 3. The mapping $\sigma^* : \mathscr{E}(\sigma(X)) \to \mathscr{E}(X)^G$ is split-surjective.

Proof. The composition $H \circ \sigma$, $H \in \mathscr{E}(\sigma(X))$ is as in [4]. As in [5] lemma 3, we let $\mathscr{H}(X)_d^G$ denote the space of homogeneous invariant fields of degree d (this makes sense, namely for each $x \in X$) and $\mathscr{W}(\sigma(X))_d$ the space of weighted homogeneous Whitney fields of degree d on $\sigma(X)$. Now put $\bar{\eta}_m = T^m_{\sigma(X)} \circ \eta \circ \Phi_m$, $m \in \mathbf{N}$, where η is chosen to split

$$\sigma^* : \mathscr{E}(\mathbf{R}^k) \to \mathscr{E}(\mathbf{R}^n)^G \quad \text{and} \quad \Phi_m$$

splits T_X^m according to proposition 2.

$$\begin{tikzcd} \mathscr{E}(\mathbf{R}^n)^G \arrow[r, shift left, "T_X^m"] \arrow[d, shift left, "\eta"] & \mathscr{E}^m(X)^G \arrow[l, shift left, "\Phi_m"] \\ \mathscr{E}(\mathbf{R}^k) \arrow[u, shift left, "\sigma^*"] \arrow[r, "T_{\sigma(X)}"'] & \mathscr{E}(\sigma(X)) \arrow[u, "\sigma_m^*"'] \end{tikzcd}$$

Evidently $\bar{\eta}_m$ splits σ_m^* by the commutative diagram. Then one derives that $\sigma_d^* : \mathscr{W}(\sigma(X)_d \to \mathscr{H}(X)_d^G$ is split-surjective, whence

$$\sigma^* = \prod_d \sigma_d^* : \prod_d \mathscr{W}(\sigma(X))_d \to \prod_d \mathscr{H}(X)_d^G$$

is split-surjective (the rings are graded via σ). Note that the resulting map $\bar{\eta}$ is continuous using the topology on $\mathscr{E}(X)$ given in [1].

It was tacitly used that $H \circ \sigma$ is Whitney if H is (noted in [4]).

Proof of the Theorem. We trace the proof in [1], with the necessary modifications.

It suffices to prove the theorem for $X = K$ compact, because we have G-invariant continuous partitions of unity on X, using Av_G.

Take the Whitney partition of unity $\{ \Phi_i \mid i \in I \}$ on $\mathbf{R}^n \backslash K$ from [1] and put $\tilde{\Phi}_i = Av_G \Phi_i$, $i \in I$. Then the family $\{ \tilde{\Phi}_i \mid i \in I \}$ of functions in $\mathscr{E}(\mathbf{R}^n \backslash K)^G$ has the properties

i′) $\{ \operatorname{supp} \tilde{\Phi}_i \mid i \in I \}$ is a locally finite family and $N(x) \leqslant N(n, G)$, a number depending on G and n, but not on x.

ii′) $\tilde{\Phi}_i \geqslant 0$ for all $i \in I$. $\sum_{i \in I} \tilde{\Phi}_i(x) = 1$ for all $x \in \mathbf{R}^n \backslash K$.

iv′) There exists a constant $\tilde{c}_k$, depending only on k, n and G, such that for all $x \in \mathbf{R}^n \backslash K$

$$\left| D^k \tilde{\Phi}_i(x) \right| < \tilde{c}_k (1 + d(x, K))^{-|k|}$$

Here ii′) is clear by the properties of Haar measure.

For all $g \in G$, induction and the chain rule shows that

$$D^k (\Phi_i \circ g)(x) = L_g^k(\Phi_i) \circ g(x),$$

where L_g^k is a linear partial differential operator of order $|k|$ with coefficients depending polynomially on g. If C_G^k is the supremum of all these coefficients over G, and N_k their number, then the definition $\tilde{\Phi}_i(x) = \int_G \Phi_i \circ g(x)\, d\mu(g)$ and the inequality valid for Φ_i shows that

$$\left| D^k \tilde{\Phi}_i(x) \right| \leqslant N_k C_G^k C_k (1 + d(x, K))^{-|k|},$$

because we have $d(g \,.\, x, K) \geqslant d(x, K)$ for all $g \in G$ (g is orthogonal). This proves iv′).

i′) is proved by induction on $\dim G$. It is evident for $\dim G = 0$ (G being then discrete, hence finite), because $\operatorname{supp} \tilde{\Phi}_i \subset G . \operatorname{supp} \Phi_i$.

Suppose the claim is true for all $p < \dim G$. Taking any $x \in \mathbf{R}^n$, the slice theorem (see [2], p. 308) enables us to look at a G-invariant neighborhood of x as being of the form $G x_H V$, $H = G_x$ the isotropy group at x, $V = V_x$ the normal space to the orbit $G(x)$ with orthogonal H-action.

There is a trivial isomorphism $\mathscr{E}(G x_H V)^G \xrightarrow{\sim} \mathscr{E}(V)^H$ (see e.g. [7], p. 51), so that each restriction $\tilde{\Phi}_i \mid G x_H V$ may be looked upon as an H-invariant function on V. Assuming that $G(x)$ is not discrete, hence $\dim H < \dim G$, the induction can be carried out.

[The inequality iii) has no counterpart; one may show $6\, d\,(\operatorname{supp} \tilde{\Phi}_i, K) + \delta \geqslant \operatorname{diam}(\operatorname{supp} \tilde{\Phi}_i)$, $\delta = \operatorname{diam} G\,(a_i)$ for an $a_i \in K$ realizing the distance to $\operatorname{supp} \Phi_i$].

Now we define $f = \tilde{H}\,(\xi) \in \mathscr{E}\,(\mathbf{R}^n)$ in a manner similar to the one in [1], by

$$f\,(x) = F^0\,(x)\,, \quad x \in K$$

$$f\,(x) = \sum_{i \in I} \tilde{\Phi}_i\,(x)\,(Av_G\, H_{a_i})\,(\xi)\,(x)\,, \quad x \notin K$$

The a_i are chosen as in [1] (note that $d\,(K, \operatorname{supp} \Phi_i) = d\,(K, \operatorname{supp} \tilde{\Phi}_i)$). $Av_G\, H_{a_i}$ and $\tilde{\Phi}_i$ being respectively equivariant and invariant, $\tilde{H}$ becomes equivariant according to (3).

We now show that the continuous linear maps $Av_G\, H_a$ are pointwise lifts, that is, still fulfil a′) and b′) (c now depending on G too)

For $a \in K$, $\xi \in E$ we have

$$\begin{aligned} T_K\,(Av_G\, H_a)\,(\xi)\,(a) &= T_K \textstyle\int_G g^{-1} \,.\, H_a\,(g \,.\, \xi)\, dg\,(a) \\ &= \textstyle\int_G g^{-1} \,.\, T_K\big(H_a(g \,.\, \xi)\big)\,(a)\, dg = \int_G g^{-1} \,.\, H\,(g \,.\, \xi)\,(a)\, dg \\ &= \textstyle\int_G H\,(\xi)\,(a)\, dg = H\,(\xi)\,(a)\,, \end{aligned}$$

using a′) and the fact that H and T_K are equivariant (prop. 1). $dg = d\mu\,(g)$ here.

Now, to tackle b′), one first observes that, as is well known, one may (without loss) assume the family $\|\,.\,\|_{\lambda \in \Lambda}$ defining the locally convex topology on E, to be upward filtering. (2′) can still be assumed to hold, and the continuity of an operator $p : E \to E$ means that to each $\|\,.\,\|_\lambda$ there is a constant c and a $\mu \in \Lambda$ such that

$$\|\, p\,(\xi)\,\|_\lambda \leqslant c\, \|\, \xi\,\|_\mu \quad \text{for all} \quad \xi \in E\,.$$

Take $\lambda \in \Lambda$. Then for all $\xi \in E$ one has an inequality

$$(x) \quad \|\, g \,.\, \xi\,\|_\lambda \leqslant c'\, \|\, \xi\,\|_{\lambda'}$$

for some $\lambda' \in \Lambda$ and a constant $c' = c'\,(G, \lambda)$, but *independent* of $g \in G$.

In fact, given $\|\,.\,\|_\lambda$ and $\varepsilon = 1$, there is a neighborhood U of e in G and $\lambda'' \in \Lambda$, $\delta > 0$ such that $g \in U$, $\|\, \xi\,\|_{\lambda''} \leqslant \delta$ implies $\|\, g \,.\, \xi\,\|_\lambda \leqslant 1$, by the continuity of the action. Let G be covered by finitely many left-translates $g_j U$, $j \in J$. To each g_j, viewed as an element of $GL\,(E)$, there is a constant c_j and $\lambda_j \in \Lambda$ such that $\|\, g_j \,.\, \xi\,\|_{\lambda''} \leqslant c_j\, \|\, \xi\,\|_{\lambda_j}$ for all $\xi \in E$. Put $c = \sup_{j \in J} c_j$ and choose $\lambda' \in \Lambda$ such that $\|\,.\,\|_{\lambda'} \geqslant \|\,.\,\|_{\lambda_j}$, $j \in J$.

Now, given $g \in G$, $\xi \in E$, we put $k = \| \xi \|_{\lambda'}$. It may be assumed that $k > 0$. Let $h = ck/\delta$, then $\| h^{-1} g_j \xi \|_{\lambda''} \leqslant \delta$ for all $j \in J$. Furthermore, $gg_j^{-1} \in U$ for some $j \in J$. Hence $\| g \, . \, \xi \|_\lambda = h \| (gg_j^{-1})(h^{-1} g_j \xi) \|_\lambda \leqslant h = (c\delta^{-1}) \| \xi \|_{\lambda'}$.

Next, using (x), (2′), the estimates of (iv′) and the Haar integral, one gets

$$\begin{aligned}
| Av_G \, H_a(\xi) |_m^L &= \sup_{\substack{x \in L \\ |k| \leq m}} \left| D^k \int_G g^{-1} \, . \, H_a(g \, . \, \xi)(x) \, dg \right| \\
&= \sup_{\substack{x \in L \\ |k| \leq m}} \left| \int_G D^k \big(H_a(g \, . \, \xi) \circ g\big)(x) \, dg \right| \\
&\leqslant \int_G \sup_{\substack{x \in L \\ |k| \leq m}} \left| [L_g^k (H_a(g \, . \, \xi))] \circ g(x) \right| dg \\
&\leqslant \int_G N_m C_G^m \, | H_a(g \, . \, \xi) |_m^{G.L} \, dg \\
&\leqslant \int_G N_m \, C_G^m \, C(m, G \, . \, L) \, \| g \, . \, \xi \|_{\lambda(m, G.L)} \, dg \\
&\leqslant C'(m, L, G) \, \| \xi \|_{\lambda'(m, L, G)}
\end{aligned}$$

for some $C' = C'(m, L, G)$ $(N_m = \sup\limits_{|k| \leq m} N_k$ etc.).

This proves b′) for $Av_G \, H_a$.

Now the rest of the proof can be carried out as in [1], replacing G_a with $Av_G \, H_a$: the evaluations (4)-(7) are valid without change and so only the *claim* of [1] with $Av_G \, H_a$ substituted for G_a must be proved. This too goes through as in [1], using a′), b′), (4)-(7) from [1] and i′), ii′), iv′).

At two points (estimation of $| S_0(x) |$ and $| S_1(x) |$) the inequality iii) is needed, and as this is a purely local matter, necessary only to obtain the inequalities $| x - a_i | \leqslant 3 | x - a |$, $| a - a_i | \leqslant 4 | x - a |$ $(x \in \mathbf{R}^n \backslash K, \; a \in K)$ the estimate iii), valid for the original Φ_i, can be used again, because we could choose a_i in supp Φ_i.

Remark. If E is a Fréchet space it is only necessary to assume that the action $G \times E \to E$ is separately continuous. Indeed, the boundedness of orbits $G \, . \, \xi$ implies via the Banach-Steinhaus theorem that $\{ \xi \mapsto g \, . \, \xi \}_{g \in G}$ is an equicontinuous set of operators, hence for a $\delta > 0$ chosen as above (no U needed) one gets $\| g \, . \, \xi \|_\lambda \leqslant \delta^{-1} \| \xi \|_\lambda$, $(g \in G, \; \xi \in E)$ instead of (x).

In the remark 5 of [1] the possibility of obtaining the pointwise lifts H_a via finite map-germs $\varphi : \mathbf{R}^n \to \mathbf{R}^p$ is discussed. We point out that there are G-equivariant extensions of these theories as developed in [7]; in particular the equivariant version of the preparation theorem is true [7], p. 64-72. Thus if the X, X' and φ mentioned in that remark are invariant under the orthogonal action of G, the W' and W may already be chosen equivariant.

Similarly, it appears that there is a G-invariant version of the Stein extension theorem mentioned in remark 4 of [1]. This results from an invariant version of corollary 2 of [1], combined with the invariant Seeley extension theorem [7], p. 108. The X in the remark 4 is invariant if φ is (G acts on $\mathbf{R}^{n+1}$ by $g \,.\, (x, y) = (g \,.\, x, y)$ for $g \in G$).

An alternative approach would be via proposition 2 and the techniques of [4], which are somewhat similar.

We also wish to point out that there is a *G-invariant version of the Whitney spectral theorem* (see [6], ch. V or [9], ch. V):

Let $\Omega \subset \mathbf{R}^n$ be open and invariant under an orthogonal action of G, G compact Lie. Let $I \subset \mathscr{E}(\Omega)^G$ (using the action (3)) be an ideal. Then $f \in \mathscr{E}(\Omega)^G$ belongs to $\bar{I}$ if and only if for each $a \in \Omega$ there is a $g_a = g \in I$ such that $J_a(g) = J_a(f)$.

This goes via a fundamental lemma [6], p. 91, for the case $\Omega = a$ cube L. With the notations of that lemma, if L is replaced by $G \,.\, L$, K by $G \,.\, K$ and T_b^m by $Av_G\, T_b^m$, then $F \in \hat{I}$ may be assumed invariant on $G \,.\, L$, whence $| \tilde{\Phi} F - f |_{G.L}^m < \varepsilon$, ($\tilde{\Phi} = Av_G\, \Phi$) can be achieved. Then one proceeds. In the more general situation considered in [9], one needs [7], lemma 1.4.1 (p. 106).

The action (4) is adapted to the operators D^k. One might consider the simpler action on $J(X)$ ($X \subset \mathbf{R}^n$ G-invariant), given by $g \,.\, F = (F^k \circ g^{-1})_{k \in \mathbf{N}^n}$, for $F = (F^k)_{k \in \mathbf{N}^n}$, $g \in G$. The corresponding problem of finding f with $J(f) = F$, given $F \in \mathscr{E}(X)^G$, is now wholly different as simple examples show (e.g. $G = \mathbf{Z}_2$ acting by reflexion in $0 \in \mathbf{R}$). If f exists at all, it must have strong singularities on K. As may be gleaned from [3], there are topological restrictions on K, depending on G. It would perhaps be feasible to obtain some answers if new operators are used instead of the D^k.

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