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and $P = \cap I_f$, where the intersection is taken over all positive functionals on A . The algebra A is called P -commutative if $xy - yx \in P$ for all x, y in A . Tiller establishes the following two theorems relating properties of the spectral radius to P -commutativity:

- (1) Let A be a Banach *-algebra which is symmetric and P -commutative. Then if $x, y \in A$, $\rho(xy) \leq \rho(x)\rho(y)$ and $\rho(x+y) \leq \rho(x) + \rho(y)$.
- (2) Let A be a Banach *-algebra with bounded approximate identity. If $\rho(x^*x) \leq \rho(x)^2$ for every x in A , then A is P -commutative.

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