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A surprisingly simple counterexample to Conjecture 2 was published by Aupetit in 1978 [15]. He makes extensive use of results obtained by Ackermans [1] in which the Gelfand representation for a commutative Banach algebra B is lifted to the matrix algebra with entries in B. Let U be the open unit disk in $\mathbf{C}$ and B the commutative Banach algebra of continuous functions on $\bar{U} \times \bar{U}$ which are holomorphic in $U \times U$. In the algebra of 2×2 matrices with entries in B define the norm by

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max \{ \|a\| + \|b\|, \|c\| + \|d\| \} .$$

Let A be the closed noncommutative subalgebra with 1 formed by the matrices

$$m = \begin{pmatrix} f(z_1, z_2) & g(z_1, z_2) \\ (z_1 + z_2) Tg(z_1, z_2) & Tf(z_1, z_2) \end{pmatrix} \text{ where } f, g \in B ,$$

and T is the isometric automorphism of B defined by $Tf(z_1, z_2) = f(z_2, z_1)$. According to Ackermans [1, Th. 3.1, 3.2] the spectrum is continuous on A. If $m \in A$ is quasi-nilpotent then again by [1, Th. 2.2] $\{0\} = \bigcup_{\phi \notin \hat{B}} \sigma(\tilde{\phi}(m))$ where $\hat{B}$ is the set of multiplicative linear functionals on A and

$$\tilde{\phi}(m) = \begin{pmatrix} \phi(f(z_1, z_2)) & \phi(g(z_1, z_2)) \\ \phi((z_1 + z_2) Tg(z_1, z_2)) & \phi(Tf(z_1, z_2)) \end{pmatrix}$$

Thus $\tilde{\phi}(m)$ is quasi-nilpotent for each $\phi \in \hat{B}$ and its square is zero by the Cayley-Hamilton Theorem. Since B is semi-simple, $m^2 = 0$. In particular $f(z_1, z_2)^2 + (z_1 + z_2) g(z_1, z_2) g(z_2, z_1) \equiv 0$, which in turn implies that $f(z_1, z_2) \equiv g(z_1, z_2) \equiv 0$. Hence $m = 0$ so there are no nonzero quasi-nilpotents in A.

6. Commutativity and the spectral radius

We now consider some weaker conditions on the spectral radius which influence the commutativity of a Banach algebra. Two familiar properties of the norm, subadditivity and submultiplicativity, are also satisfied by the spectral radius on *commuting* elements. Does the imposition of these properties on the whole algebra then imply commutativity? Because

$$| x |_\sigma = \sup \{ | \lambda | : \lambda \in \sigma(x) \}$$

vanishes for every x in the radical, $| \cdot |_\sigma$ certainly satisfies both properties in a noncommutative radical algebra. Consequently commutativity modulo the radical is the most that could be expected. In 1971, Bernard Aupetit [7] announced that in a Banach algebra A each of these conditions separately imply that $A/\mathrm{Rad}(A)$ is commutative and that in fact the following are equivalent:

(1) $| x + y |_\sigma \leqslant \alpha (|x|_\sigma + |y|_\sigma)$ for some $\alpha > 0$.
("$| \cdot |_\sigma$ is subadditive")

(2) $| xy |_\sigma \leqslant \beta | x |_\sigma | y |_\sigma$ for some $\beta > 0$. ("$| \cdot |_\sigma$ is submultiplicative")

(3) $A/\mathrm{Rad}(A)$ is commutative (A is "almost commutative".)

Proofs of these equivalences were published later [9] using subharmonic functions. Several additional equivalent conditions involving the spectral radius and spectral diameter were added at this time, the most significant of which is

(4) $| \cdot |_\sigma$ is uniformly continuous on A.

In the case of algebras with identity Aupetit was in fact able to restrict the subadditive and submultiplicative conditions to a neighborhood of the identity.

Using a more elementary algebraic approach J. Zemánek has also established the equivalence of conditions (1) - (4). An account of this first appeared in a joint paper with V. Pták [70] where three auxiliary conditions were introduced, each equivalent to (1) - (4). A more refined and comprehensive treatment has now been given by Zemánek [112]. This version rests on an analysis of the pseudonorm

$$s(x) = \sup \{ | x - u^{-1} x u |_\sigma : u \text{ is invertible in } A \text{ or } A_1 \},$$

and depends on an extension of the Jacobson density theorem given in A. M. Sinclair's monograph [79, p. 36]. In [112] Zemánek also established related results involving the spectral radius and used the techniques developed there to prove analogous theorems for real Banach algebras. His work renders the equivalences (1) - (4) accessible without reference to potential theory and subharmonic functions while sacrificing some of the sharpness of Aupetit's results. We now give proofs of these equivalences following the arguments of Pták-Zemánek [70], but supressing the auxiliary conditions mentioned above.

First we note that (3) implies both (1) and (2). This follows immediately from the behavior of $|\cdot|_\sigma$ on the quotient A/R, where $R = \mathrm{Rad}\,(A)$.

LEMMA 6.1. *For any x in the Banach algebra A, $|x+R|_\sigma = |x|_\sigma$.*

Proof. We show that $\sigma(x+R) \cup \{0\} = \sigma(x) \cup \{0\}$. Let $\lambda \neq 0$ belong to $\sigma(x)$ so that x/λ is quasi-singular in A. If $x/\lambda + R$ were quasi-regular in A/R, then $x/\lambda \circ y \in R$ for some y in A. Since every element of R is quasi-regular, x/λ would be quasi-regular also. Thus $\lambda \in \sigma(x+R)$. The reverse inclusion is similarly proved and the result clearly extends to any quasi-regular ideal.

PROPOSITION 6.2. *If A is an almost commutative Banach algebra, then $|\cdot|_\sigma$ is subadditive and submultiplicative.*

Proof. $|x+y|_\sigma = |x+y+R|_\sigma \leqslant |x+R|_\sigma + |y+R|_\sigma = |x|_\sigma + |y_\sigma|$. Analogously $|xy|_\sigma \leqslant |x|_\sigma \cdot |y|_\sigma$.

PROPOSITION 6.3. *If $|\cdot|_\sigma$ is subadditive, then $|\cdot|_\sigma$ is uniformly continuous.*

Proof. The subadditivity of $|\cdot|_\sigma$ yields $||x|_\sigma - |y|_\sigma| \leqslant |x y|_\sigma \leqslant \|x-y\|$, which actually shows Lipschitz continuity with a constant of 1.

PROPOSITION 6.4. *If $|\cdot|_\sigma$ is subadditive, then $|\cdot|_\sigma$ is submultiplicative.*

Proof. We may certainly assume $\alpha \geqslant 1$. Let $\beta = 9\alpha^2$. To show that $|xy|_\sigma \leqslant 9\alpha^2 |x|_\sigma |y_\sigma|$ it suffices to choose $\lambda \in \mathbf{C}$ such that $|\lambda| > 9\alpha^2 |x|_\sigma |y|_\sigma$ and show $\lambda - xy$ is invertible. (We now adjoin an identity to A if it has none, but (2) is assumed to hold only in A). Choose complex numbers μ and ν satisfying $\mu\nu = \lambda$, $|\mu| > 3\alpha |x|_\sigma$, $|\nu| > 3\alpha |y|_\sigma$. Put $u = x/\mu$ and $v = y/\mu$. Then $|u|_\sigma < 1/3\alpha$ and $|v|_\sigma < 1/3\alpha$. Since u commutes with $(1-u)^{-1}$ we have $|(1-u)^{-1} u|_\sigma \leqslant |(1-u)^{-1}|_\sigma |u|_\sigma < [1/(1-1/3\alpha)]\,[1/3\alpha] = 1/3\alpha - 1 \leqslant 1/2\alpha$ since $\alpha > 1$. Similarly $|(1-v)^{-1}v|_\sigma < 1/2\alpha$. Now $\lambda - xy$ is invertible if $1 - uv$ is; but $1 - uv = (1-u)\,[1+(1-u)^{-1} u + v(1-v)^{-1}]\,(1-v)$ where each factor is invertible, the middle one because $|(1-u)^{-1} u + v(1-v)^{-1}|_\sigma \leqslant \alpha |(1-u)^{-1} u|_\sigma + \alpha |v(1-v)^{-1}|_\sigma < 1$.

PROPOSITION 6.5. *If $|\cdot|_\sigma$ is submultiplicative, then $|\cdot|_\sigma$ is subadditive.*

Proof. It is convenient to consider the cases with and without identity separately. Suppose A has an identity and $|\lambda| > \beta |x|_\sigma + \beta |y|_\sigma$ (assume

$\beta \geqslant 1$). Then $\lambda - x$ is invertible and $\lambda - (x+y) = (\lambda - x)\left[1-(\lambda-x)^{-1}y\right]$ while $|(\lambda-x)^{-1}y|_\sigma \leqslant \beta\,|(\lambda-x)^{-1}|_\sigma\,|y|_\sigma \leqslant \beta\,(|\lambda|-|x|_\sigma)^{-1}\,|y|_\sigma < 1$. Thus $\lambda - (x+y)$ is invertible and the conclusion follows with $\alpha = \max\{\beta, 1\}$. If A has no identity, again assume $\beta \geqslant 1$ and that $|\lambda| > \beta\,|x|_\sigma + \beta\,|y|_\sigma$. Since $\lambda - x$ is invertible in A_1, we have $(\lambda-x)^{-1} = v + u$ where $v \in A$ and $\mu \in \mathbf{C}$. From $(\lambda-x)(\mu+v) = 1$ we have $\mu = 1/\lambda$, and hence $v = (\lambda-x)^{-1} - 1/\lambda$. Now $\lambda - (x+y) = (\lambda-x)\left[1-(\lambda-x)^{-1}y\right] = (\lambda-x)\left[1-(1/\lambda y - vy\right] = (\lambda-x)\left[1-vy\left(1-(1/\lambda)y\right)^{-1}\right]\left(1-(1/\lambda)y\right)$, where $1-(1/\lambda)y$ is invertible since $|\lambda| > |y|_\sigma$. Since A is an ideal in A_1, we have $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|v|_\sigma\,|y\,(1-(1/\lambda)y)^{-1}|_\sigma$. But

$$|v|_\sigma \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda|-|x|_\sigma)} \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda|-\beta|x_{\cdot\sigma})}$$

and $|y\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant |\lambda| \cdot |y|_\sigma/(|\lambda|-|y|_\sigma) \leqslant |\lambda| \cdot |y|_\sigma/(|\lambda|-\beta|y|_\sigma)$. Multiplying these estimates we obtain $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|x|_\sigma\,|y|_\sigma/(|\lambda|-\beta|x|_\sigma)(|\lambda|-\beta|y|_\sigma) < 1$. So again we may take $\alpha = \max\{\beta, 1\}$.

PROPOSITION 6.6. *If* $|\cdot|_\sigma$ *is subadditive on* A, *then* A *is almost commutative.*

Proof. Since $|\cdot|_\sigma$ must be submultiplicative, Corollary 5.5 (to the results of Hirschfeld-Żelazko) states that $A/\mathrm{Rad}\,(A)$ must be commutative.

It is of interest that the uniform continuity of $|\cdot|_\sigma$ implies its Lipschitz continuity. It is easy to see that the Lipschitz constant can be taken as $1/\varepsilon$ where $\|x - y\| < \varepsilon$ implies $\left\|x|_\sigma - |y|_\sigma\right| \leqslant 1$. We have already seen that the subadditivity of $|\cdot|_\sigma$ implies its Lipschitz continuity.

7. FURTHER GENERALIZATIONS AND RELATED RESULTS

During the past forty years the general subject of this paper has received attention from many authors. Our purpose here is to give a brief discussion of some of the relevant literature.

Ramaswami studies in [71] the Mazur-Gelfand theorem under minimal hypothesis. He weakens the associative law and also the triangle inequality; the former in several ways. In all he gives six different sets of sufficient conditions for a "generalized" complex Banach algebra to coincide with the complex field. He treats real Banach algebras in the same spirit.

Elementary proofs of the Mazur-Gelfand theorem which avoid direct appeal to complex function theory (in particular to Liouville's theorem)