4. Norm conditions and topological divisors of zero

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **26 (1980)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

[68, pp. 158-163]) by using the nonemptiness of the spectrum instead of finite-dimensionality to obtain the first major step. The remainder of the proof, though long, consists of entirely elementary algebraic verifications. We shall not reproduce this proof here.

If A is an algebra over $\mathbf{R}$, its complexification $A_{\mathbf{C}}$ is analogous to the construction of $\mathbf{C}$ from $\mathbf{R}$. (We may think of $A_{\mathbf{C}}$ as $A+iA$.) $A_{\mathbf{C}}$ will have a unit if and only if A does. Moreover if A is a normed algebra, the norm may be extended to $A_{\mathbf{C}}$ in a standard fashion (*vid.* Rickart [73, pp. 8-9]) so that the extension is complete whenever the original norm on A is. The spectrum $\sigma(x)$ of an element x in A is defined to be its spectrum in $A_{\mathbf{C}}$. Thus if A has a unit e, $\alpha + i\beta \in \sigma(x)$ if and only if the element $(\alpha + i\beta)(e, 0) - (x, 0)$ is singular in $A_{\mathbf{C}}$. Again by analogy with the complex numbers it is immediate that if a and b are *commuting* elements of A, (a, b) is invertible in $A_{\mathbf{C}}$ if and only if $a^2 + b^2$ is invertible in A. Thus $\alpha + i\beta \in \sigma(x)$ if and only if $(\alpha - x)^2 + \beta^2$ is singular in A.

4. Norm conditions and topological divisors of zero

In his original paper Mazur [57] also announced a companion theorem.

THEOREM 4.1. [Mazur]. *A real normed algebra A satisfying $\|xy\| = \|x\| \|y\|$ is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

It is particularly worth noting that Theorem 4.1 (as stated in Mazurs' paper) carries no assumption that A has an identity element. Our goal in this section is to prove a generalization of this theorem due to Irving Kaplansky [53]. Its formulation depends on the concept of a topological divisor of zero in a normed algebra introduced by Shilov in 1940 [77]. An element x of a normed algebra is said to be a *topological divisor of zero* (t.d.z.) if there is a sequence y_n, $\|y_n\| = 1$, such that $xy_n \to 0$ or $y_n x \to 0$. Kaplansky's result is then:

THEOREM 4.2. [Kaplansky]. *If A is a real normed algebra having no nonzero topological divisors of zero, then A is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

The development of the proof below closely follows Kaplansky's line of reasoning except for changes made to avoid the use of algebraic results not established here and instead to take advantage of Theorem 2.1. Mazur's original proof of Theorem 4.1 used algebraic results analogous to some facts of algebraic number theory and a form of Frobenius' theorem not assuming an identity element. Here, however we will adhere as closely as possible to techniques in the spirit of the theory of normed algebras.

The recurrent theme of the nonemptiness of the spectrum of any element of a complex normed algebra will again surface in dealing with algebras not necessarily having an identity. Consequently a suitable definition of the spectrum is required for algebras without identity, in which every element is *a fortiori* singular. In any ring define a binary operation $x \circ y = x + y - xy$. It is easily verified that $\circ$ is associative and has 0 as a two-sided identity. An element y, necessarily unique, is called the *quasi-inverse* of x if $x \circ y = 0 = y \circ x$. If y exists, x is said to be *quasi-regular*, and its quasi-inverse is denoted x'. If no such y exists, x is *quasi-singular*. The set of quasi-regular elements of a ring A is then a group under the circle operation. If A has an identity element 1, the relation $1 - (x \circ y) = (1-x)(1-y)$ shows that x is quasi-regular if and only if $1 - x$ is invertible. Guided by these observations, it is standard to formulate a definition of spectrum without reference to an identity. If $x \in A$, an algebra over $\mathbf{C}$, a nonzero complex number λ belongs to the *spectrum* $\sigma(x)$ of x if and only if x/λ is quasi-singular. The spectrum will contain 0 unless A has an identity and x is invertible. For real algebras the spectrum is defined as before via the complexification, whence a nonzero complex number $\alpha + i\beta \in \sigma(x)$ if and only if $(2\alpha x - x^2)/(\alpha^2 + \beta^2)$ is quasi-singular. It is easy to check that this definition coincides with the original one when A has an identity.

We now survey some basic properties pertaining to the concepts just introduced. In what follows A denotes a complete normed algebra over $\mathbf{R}$ or $\mathbf{C}$ unless the contrary is stated.

PROPOSITION 4.3. *Every x in A satisfying $\|x\| < 1$ is quasi-regular with $x' = -\sum_{n=1}^{\infty} x^n$ and $\|x'\| \leqslant \|x\|/(1-\|x\|)$.*

Proof. The geometric series $\sum_{n=1}^{\infty} x^n$ converges by completeness and $x - \sum_{n=1}^{\infty} x^n + x \sum_{n=1}^{\infty} x^n = 0$. The bound on $\|x'\|$ follows by applying the triangle inequality to $x' = -x + xx'$.

PROPOSITION 4.4. *If y is quasi-regular, so is $y + x$ for $\|x\| < k = 1/(1+\|y'\|)$, and $\|(y+x)' - y'\| \leqslant \|x\|/(k-\|x\|)\,k$.*

Proof. If $\| x \| < k$, $\| x - xy' \| \leqslant \| x \| (1+\|y'\|) < 1$, so $u = x - xy'$ is quasi-regular. Since $(y+x) \circ y' = u$, $y' \circ u'$ is a right quasi-inverse for $y + x$. Repeating the argument for $x - y'x$ we find that $y + x$ also has a left quasi-inverse. Thus $y + x$ is quasi-regular and $(y+x)' = y' \circ u'$. Moreover $(y+x)' - y' = y' \circ u' - y' = u' - y'u'$, so $\| (y+x)' - y' \| \leqslant (1+\|y'\|) \| u' \| \leqslant (1+\|y'\|) \| u \|/(1-\|u\|) \leqslant \| x \| (1+\|y'\|)^2/(1-\|x\| (1+\|y'\|)) = \| x \|/(k-\|x\|)\, k$.

Corollary 4.5. (a) *The set* QR *of quasi-regular elements of* A *is open and* $y \to y'$ *is continuous.*

(b) *If* A *has an identity, the set* G *of invertible elements is open and* $y \to y^{-1}$ *is continuous.*

Proof. Part (a) follows immediately from Propositions 4.3 and 4.4. Since $G = 1 - QR$ and $y^{-1} = 1 - (1-y)'$ for $y \in G$, (b) follows from (a).

Corollary 4.6. *The spectrum of any element is closed and bounded.*

Proof. If $\lambda \in \sigma(x)$, $\lambda \neq 0$, x/λ is quasi-singular so $\| x/\lambda \| = | \lambda |^{-1} \| x \| \geqslant 1$ by Proposition 4.3. Thus $| \lambda | \leqslant \| x \|$ for all $\lambda \in \sigma(x)$. If $\lambda \notin \sigma(x)$, $\lambda \neq 0$, then x/λ is quasi-regular. Since QR is open x/μ will be quasi-regular for all μ sufficiently near λ. If $0 \notin \sigma(x)$, then $x \in G$, so $\lambda - x \in G$ for all λ sufficiently near 0. Thus $\mathbf{C} \setminus \sigma(x)$ is open and $\sigma(x)$ is closed.

Corollary 4.7. *If* y *is quasi-regular, then* $[(y+\lambda y)'-y']/\lambda \to (y')^2 - y'$ *as* $\lambda \to 0$.

Proof. Taking $x = \lambda y$ in Proposition 4.4 so that $u = \lambda(y-yy') = -\lambda y'$ we have $[(y+\lambda y)' - y']/\lambda = [(-\lambda y')' - y'(-\lambda y')']/\lambda = \sum_{n=1}^{\infty} (-1)^n \lambda^{n-1} [(y')^n - (y')^{n+1}] \to (y')^2 - y'$ as $\lambda \to 0$.

Even if A has no identity element we may speak of $\lambda - x$ being a topological divisor of zero for any scalar λ, i.e., $\lambda y_n - xy_n \to 0$ or $\lambda y_n - y_n x \to 0$ for a sequence y_n, $\| y_n \| = 1$. With this convention we have the following result:

Proposition 4.8. (a) *If* x *belongs to the frontier of* QR, *then* $1 - x$ *is a t.d.z.*

(b) *If* A *has an identity and* x *belongs to the frontier of* G, *then* x *is a t.d.z.*

Proof. (a) Since QR is open, x is quasi-singular and $x_n \to x$ where $x_n \in QR$. Now $\{x_n'\}$ is not bounded; for otherwise $x \circ x_n' = (x-x_n) - (x-x_n)\, x_n'$ implies that $\| x \circ x_n' \| < 1$ and hence $x \circ x_n' \in QR$ for large n. This would mean $x \in QR$. Let $z_n = x_n'/\| x_n' \|$. Then $z_n - xz_n = (x-x_n)/\| x_n' \| - (x-x_r)\, z_r - x/\| x_n' \| \to 0$. Part (b) follows from (a) since $G = 1 - QR$.

PROPOSITION 4.9. *Suppose A is a real normed algebra. If x is a t.d.z. in the completion or complexification of A, then x is a t.d.z. in A.*

Proof. In the case of the complexification we suppose x is a t.d.z. in A_{C}, so we have, say, $x\,(a_n + ib_n) \to 0$ where $a_n, b_n \in A$ and $\| a_n + ib_n \| = 1$. Then $xa_r \to 0$ as $xb_r \to 0$. Not both $\{a_n\}$ and $\{b_n\}$ can converge to zero, for then $a_n + ib_n \to 0$. If $a_n \to 0$, by taking subsequences if necessary and normalizing we can obtain $xa_n \to 0$ where $\| a_n \| = 1$.

In the case of the completion suppose x is a t.d.z. in the completion A' of A. Then $xy_n \to 0$ for $y_n \in A'$ and $\| y_n \| = 1$. Using the density of A in A' and constructing a suitable "diagonal" sequence we can choose $z_n \in A$ such that $xz_r \to 0$ and $z_n \to 0$. It follows that x is a t.d.z. in A.

The following lemma, crucial to the proof of Theorem 4.2, employs the same technique Gelfand used to prove the Mazur-Gelfand theorem to establish a necessary condition for the spectrum of an element to be "degenerate", i.e., $\{0\}$.

LEMMA 4.10. *If $\sigma(x) = \{0\}$, then x is a t.d.z.*

Proof. We may, of course, assume that $x \neq 0$, and by Proposition 4.9 it suffices to consider the complex case. Since $\lambda x \in QR$ for all $\lambda \in \mathbf{C}$ we can consider $\{(\lambda x)' : \lambda \in \mathbf{C}\}$. This set must be unbounded. Otherwise for any continuous linear functional f on A, $\lambda \to f\ [(\lambda x)']$ would be bounded and by Corollary 4.7 entire. Liouville's theorem implies that f must be constant. Therefore $f\ (0) = f\ (1)$, i.e., $0 = f\ (x')$. Since f was arbitrary the Hahn-Banach theorem implies that $x' = 0$, so $x = 0$, contradicting our assumption. Thus we may choose a complex sequence $\lambda_n \to \infty$ such that $y_n = (\lambda_n x)'$ is unbounded. Letting $z_r = y_n/\| y_n \|$ we have $xz_n = y_n x y_n/\lambda_n \| y_n \| = x/\| y_n \| + z_n/\lambda_n \to 0$. Thus x is a t.d.z.

By examining the proof of Corollary 4.7 in the case where it is applied above and by using the "completion" part of Proposition 4.9 one can omit the completeness of A from the hypothesis of Lemma 4.10. We shall, however, not need this in the proof of Theorem 4.2 to which we now turn.

Proof of Kaplansky's Theorem (4.2). The crux of the proof is to show that A must have an identity element. Then it is easy to see that A is a division algebra so that Mazur's Theorem 2.1 applies. Let x be any nonzero element of A and let B be the complexification of the completion of A. Since x is not a t.d.z. in B, $\sigma(x)$ contains nonzero elements; it is also closed and bounded. Therefore we may choose $\lambda = \alpha + i\beta \neq 0$ belonging to the frontier of $\sigma(x)$. Then $z = (2x - x^2)/(\alpha^2 + \beta^2)$ is a quasi-singular element which is the limit of quasi-regular elements. By Proposition 4.8 $1 - z$ is a t.d.z. in B, hence in A. For each y in A it follows that $y - yz$ and $y - zy$ are t.d.z.; thus $yz = y = zy$ for all y in A. So A has an identity element. The set $S = \{x \in A : x \neq 0 \text{ and } x \text{ is singular}\}$ is open since S does not meet the frontier of G. Then $A \setminus \{0\} = G \cup S$, where $G \cap S = \varnothing$, so either $A \setminus \{0\}$ is disconnected or $S = \varnothing$. The former condition occurs only if $\dim_{\mathbf{R}} A = 1$ [see Lemma 4.12 below for details.] Then $A \simeq \mathbf{R}$ and the conclusion holds. The latter immediately implies that A is a division algebra, and Theorem 2.1 concludes the proof.

Since a normed algebra satisfying $\| xy \| = \| x \| \, \| y \|$ can have no non-zero topological divisors of zero, Theorem 4.2 implies Theorem 4.1. The hypothesis of 4.1 can be weakened slightly:

COROLLARY 4.11. *If A is a real normed algebra satisfying $\| xy \| \geqslant \beta \| x \| \, \| y \|$ for some positive constant β and all x, y, then A is isomorphic to* **R**, **C**, *or* **H**.

If the multiplicative norm condition $\| xy \| = \| x \| \, \| y \|$ holds for all x, y in a Banach algebra A with unit e and $\| e \| = 1$, then certainly $1 = \| xx^{-1} \| = \| x \| \, \| x^{-1} \|$, i.e., $\| x^{-1} \| = \| x \|^{-1}$ for all invertible x in A. R. E. Edwards [37] has shown that in a real Banach algebra with identity this condition also characterizes the classical division algebras over **R**. This follows in an elementary way from Mazur's Theorem 2.1. We begin with two lemmas both of which are well-known.

LEMMA 4.11. *Let A be a Banach algebra with identity e and G the group of invertible elements of A. If $\{x_r\} \subset G$, $\| x_n^{-1} \|$ is a bounded sequence, and $x_n \to x$, then $x \in G$.*

Proof. Suppose $\| x_n^{-1} \| \leqslant M$ for every n. The identity $x_n^{-1} - x_m^{-1} = x_n^{-1}(x_m - x_n)\, x_m^{-1}$ then implies that $\| x_n^{-1} - x_m^{-1} \| \leqslant M^2 \| x_n - x_m \|$; hence $\{x_n^{-1}\}$ is Cauchy. Let $y = \lim x_n^{-1}$. The continuity of multiplication gives $xy = e = yx$ so that $x \in G$.

LEMMA 4.12. *If* A *is a normed linear space with* $\dim_{\mathbf{R}} A \geqslant 2$, *then* $A \setminus \{0\}$ *is connected.*

Proof. Let a and b belong to $A \setminus \{0\}$. If a and b are independent over $\mathbf{R}$, the segment $\{ta + (1-t)b : 0 \leqslant t \leqslant 1\}$ does not contain 0. If a and b are dependent over $\mathbf{R}$, we can choose c independent of a (hence of b) over $\mathbf{R}$ since $\dim_{\mathbf{R}} A \geqslant 2$. Thus both a and b can be joined to c by a line not containing 0. This proves $A \setminus \{0\}$ is arcwise connected.

THEOREM 4.13. [Edwards]. *If* A *is a Banach algebra with identity such that* $\| x^{-1} \| = \| x \|^{-1}$ *for all invertible* x, *then* A *is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$. *If* A *is a complex Banach algebra satisfying this condition,* A *is isomorphic to* $\mathbf{C}$.

Proof. If $\dim_{\mathbf{R}} A = 1$, the conclusion is obvious. Otherwise we show that A is a division algebra and apply Mazur's theorem. According to Corollary 4.5 (b), the set G of invertible elements is open in A, hence in $A \setminus \{0\}$. We show that G is also closed in $A \setminus \{0\}$. Let $x_n \in G$ and $x_n - x \neq 0$. Then $\| x_n^{-1} \| = \| x_n \|^{-1}$. Since $\| x_n \|^{-1} \to \| x \|^{-1} > 0$, we see that $\| x_n \|^{-1}$ is a bounded sequence of real numbers. By Lemma 4.11, $x \in G$ with $x_n^{-1} \to x^{-1}$. As we are assuming $\dim_{\mathbf{R}} A \geqslant 2$, $A \setminus \{0\}$ is connected by Lemma 4.12 and $G = A \setminus \{0\}$ as required.

COROLLARY 4.14. *If* A *is a Banach algebra with identity and there is a constant* α *satisfying* $\| x \| \, \| x^{-1} \| \leqslant \alpha \| x x^{-1} \|$ *for all invertible* x, *then* A *is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

Proof. This weaker condition on the norm also guarantees that $\| x_n^{-1} \|$ is a bounded sequence.

M. Seetharama Gowda [42] has given a different proof of the complex version of Edwards' theorem based on numerical range. He requires only that the set $\{x \in A : \| x \| \, \| x^{-1} \| = 1\}$ have nonempty interior. This apparently weaker condition is also proved sufficient by S. Aurora [17] in a version of Edwards' theorem in the more general context of metric rings.

In the direction of the results of Gowda and Aurora one might study the impact of imposing various conditions dealt with in this survey on a nonempty open subset of the algebra rather than on the whole algebra. For example, $\| xy \| = \| x \| \, \| y \|$ where x, y range over an open set U, or $\| xx' \| = \| x \| \, \| x' \|$ for all x in an open subset U of $\underline{QR}$. These conditions can be considered for algebras with or without identity. As we shall see later B. Aupetit [9] has obtained some theorems of this kind.

By considering the incomplete normed algebra of polynomials with complex coefficients and norm $\| \sum_{i=0}^{n} c_i x^i \| = \sum_{i=0}^{n} | c_i |$, it is easy to see that completeness is essential in Edwards' theorem.

5. Norm conditions and commutativity

The investigation of further conditions on the norm of a normed algebra lead to consideration of the *spectral radius*, defined by

$$\rho(x) = \lim_{n \to \infty} \| x^n \|^{1/n} .$$

We list several of its properties for later reference (see [73, pp. 10, 30]). For elements x and y in a normed algebra A:

(1) $\rho(x) \leqslant \| x \|$.

(2) $\rho(xy) = \rho(yx)$ and $\rho(x^n) = \rho(x)^n$.

(3) $\rho(x+y) \leqslant \rho(x) + \rho(y)$ and $\rho(xy) \leqslant \rho(x)\rho(y)$ if $xy = yx$.

(4) $\rho(x) = \sup \{ | \lambda | : \lambda \in \sigma(x) \}$ provided A is complete.

In view of the exceptionally strong consequences of the multiplicative norm condition $\| xy \| = \| x \| \, \| y \|$ it is natural to inquire into the algebraic implications of the similar condition

$$\| x^2 \| = \| x \|^2 . \qquad (*)$$

One familiar consequence of (*) in a normed algebra A is that

$$\rho(x) = \| x \| \qquad (**)$$

for all x in A. On the other hand since ρ always satisfies $\rho(x^2) = \rho(x)^2$, the conditions (*) and (**) are equivalent in any normed algebra. Their importance can be surmised by noting that they imply the Gelfand representation for commutative Banach algebras is isometric.

Almost simultaneously Claude Le Page [54] and R. A. Hirschfeld together with W. Żelazko [47] discovered independently that (*) in fact implies the commutativity of A. Although both papers use essentially the