# NORM AND SPECTRAL CHARACTERIZATIONS IN BANACH ALGEBRAS

# NORM AND SPECTRAL CHARACTERIZATIONS IN BANACH ALGEBRAS

by V. A. Belfi and R. S. Doran

## 1. Introduction

Commutative Banach algebras enjoy a remarkably complete and beautiful structure theory due, in large part, to the efforts of a single man, I. M. Gelfand. The simplicity and beauty of this theory has prompted a number of mathematicians over the past fifteen years to seek conditions on an algebra (usually on its norm, spectral radius, etc.) in order that it be commutative.

Looking back to the late thirties, just before the splendid work of Gelfand, the question of characterizing those Banach algebras which are isomorphic to the real or complex numbers, or to the quaternions was being considered by S. Mazur. A corollary of one of his results, namely that a complex normed algebra which is a field is isomorphic to the complex numbers, turned out to be so basic that it is the foundation on which Gelfand's whole structure theory for commutative algebras rests.

Many papers have now been written concerning the characterization of commutativity in Banach algebras, and also on the problem of determining which algebras are isomorphic to **R**, **C** or the quaternions **H**. The purpose of this paper is to survey the results of these papers, giving full proofs (when possible), and also giving historical perspective. Our goal is to provide the reader, as best we are able, with an unobstructed view of the subject. To this end, and also to make the paper accessible to a wide audience, we have included a few well-known arguments from the general theory of Banach algebras. These serve, in several cases, to reveal important techniques which shed light on later developments. A substantial bibliography has been assembled to aid the reader wishing to pursue the subject further.

## 2. The Mazur-Gelfand theorem

A *normed algebra* is an associative linear algebra $A$ over the real or complex field which is also a normed linear space satisfying $\| xy \| \leqslant \| x \| \cdot \| y \|$ for every $x$ and $y$ in $A$. If $A$ is complete in this norm, it is called a *Banach algebra*.

In 1938 Stanislaw Mazur [57] announced the following classification theorem for real normed division algebras:

Theorem 2.1. [Mazur]. *A real normed algebra with identity in which every nonzero element has an inverse is isomorphic to either* **R**, **C**, *or the quaternions* **H**.

An immediate consequence of this result, which classifies normed division algebras over **C**, is known as the Mazur-Gelfand theorem:

Theorem 2.2. [Mazur-Gelfand]. *A complex normed algebra with identity in which every nonzero element has an inverse is isomorphic to the complex numbers.*

This complex version follows in a standard way from Theorem 2.1 since every complex normed algebra is also a real normed algebra, and the possibilities of **R** and **H** are easily eliminated in the complex case.

An historical precursor to Mazur's theorem was published by Alexander Ostrowski in 1918 [65]. It states that every field with an archimedean valuation is topologically isomorphic with a subfield of **C** carrying the ordinary absolute value as its valuation. If the field has additionally the structure of a real vector space, then the possibilities are further reduced to **R** or **C**.

The details of Mazur's proof were too lengthy to be included in his announcement, and it was Gelfand who furnished the first published proof [38] of the complex version, which bears his name. His proof, different from Mazur's, uses a generalized form of Liouville's theorem from complex analysis. The theorem was established independently by Lorch [55] whose proof likewise was based on Liouville's theorem; he points out that substantially the same argument was given earlier by Taylor [91]. We now record this elegant proof in a form which uses the classical version of Liouville's theorem.

*Gelfand's proof of the Mazur-Gelfand Theorem 2.2.* For any element $x$ of the complex normed algebra $A$ with identity $e$, we show that $x = \lambda e$

for some complex $\lambda$. Suppose to the contrary that $x - \lambda e \neq 0$ for all $\lambda$ in **C**. Since $A$ is a division algebra, it follows that $x - \lambda e$ is invertible for all $\lambda$, i.e., $(x-\lambda e)^{-1}$ exists. Let $x(\lambda) = (x-\lambda e)^{-1}$. By the Hahn-Banach theorem there is a bounded linear functional $L$ on $A$ such that $L(x^{-1}) = 1$. Define $g : \mathbf{C} \to \mathbf{C}$ by $g(\lambda) = L(x(\lambda))$; then $g(0) = 1$. Moreover, $g$ is an entire function. Indeed, since $x(\lambda) - x(\mu) = (\lambda-\mu)\, x(\lambda)\, x(\mu)$ for $\lambda$, $\mu$ in **C**, it follows that

$$\lim_{\lambda\to\mu} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \lim_{\lambda\to\mu} L(x(\lambda)\, x(\mu)) = L(x(\mu)^2).$$

Further $|g(\lambda)| \leqslant \|L\| \, \|x(\lambda)\|$ and since $x(\lambda) \to 0$ as $|\lambda| \to \infty$, $g(\lambda) \to 0$. By Liouville's theorem the bounded entire function $g$ is constant; hence $g \equiv 0$. This is a contradiction since $g(0) = 1$, and the proof is complete.

The *spectrum* of an element $x$ of a complex algebra with identity $e$ is the set $\sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda e \text{ is singular}\}$, so Gelfand's proof can be viewed as a demonstration that the spectrum of any element of a complex normed algebra with identity is nonempty. This fact together with the application of Liouville's theorem forms a continuous thread running through the generalizations and related results presented in this paper.

## 3. Classification of real normed division algebras

Although it does not appear to be widely known, Mazur's original paper on normed division algebras [57] considers *only* the case of algebras over **R**. If a real division algebra is also finite-dimensional, the classical theorem of Frobenius classifies it as **R**, **C** ,or **H**. Mazur demonstrated finite-dimensionality in two steps: first he used a rather lengthy argument involving analytic function theory to show that it cannot contain a subalgebra isomorphic to the rational functions in one indeterminate with real coefficients. He then quoted an algebraic theorem to the effect that every real infinite-dimensional division algebra must contain such a subalgebra. The details of the first step may now be found in W. Żelazko's book [109, pp. 18-22].

F. F. Bonsall and J. Duncan [30] have given a more direct and self-contained proof of Mazur's theorem, which relies on precisely the same analytic fact as Gelfand's proof of the complex version; namely that every element of a complex normed algebra with identity has nonempty spectrum. They modify a standard proof of Frobenius' theorem (*vid.* Pontrjagin

[68, pp. 158-163]) by using the nonemptiness of the spectrum instead of finite-dimensionality to obtain the first major step. The remainder of the proof, though long, consists of entirely elementary algebraic verifications. We shall not reproduce this proof here.

If $A$ is an algebra over $\mathbf{R}$, its complexification $A_{\mathbf{C}}$ is analogous to the construction of $\mathbf{C}$ from $\mathbf{R}$. (We may think of $A_{\mathbf{C}}$ as $A+iA$.) $A_{\mathbf{C}}$ will have a unit if and only if $A$ does. Moreover if $A$ is a normed algebra, the norm may be extended to $A_{\mathbf{C}}$ in a standard fashion (*vid.* Rickart [73, pp. 8-9]) so that the extension is complete whenever the original norm on $A$ is. The spectrum $\sigma(x)$ of an element $x$ in $A$ is defined to be its spectrum in $A_{\mathbf{C}}$. Thus if $A$ has a unit $e$, $\alpha + i\beta \in \sigma(x)$ if and only if the element $(\alpha+i\beta)(e, 0) - (x, 0)$ is singular in $A_{\mathbf{C}}$. Again by analogy with the complex numbers it is immediate that if $a$ and $b$ are *commuting* elements of $A$, $(a, b)$ is invertible in $A_{\mathbf{C}}$ if and only if $a^2 + b^2$ is invertible in $A$. Thus $\alpha + i\beta \in \sigma(x)$ if and only if $(\alpha - x)^2 + \beta^2$ is singular in $A$.

## 4. Norm conditions and topological divisors of zero

In his original paper Mazur [57] also announced a companion theorem.

THEOREM 4.1. [Mazur]. *A real normed algebra $A$ satisfying $\|xy\| = \|x\| \|y\|$ is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

It is particularly worth noting that Theorem 4.1 (as stated in Mazurs' paper) carries no assumption that $A$ has an identity element. Our goal in this section is to prove a generalization of this theorem due to Irving Kaplansky [53]. Its formulation depends on the concept of a topological divisor of zero in a normed algebra introduced by Shilov in 1940 [77]. An element $x$ of a normed algebra is said to be a *topological divisor of zero* (t.d.z.) if there is a sequence $y_n$, $\|y_n\| = 1$, such that $xy_n \to 0$ or $y_n x \to 0$. Kaplansky's result is then:

THEOREM 4.2. [Kaplansky]. *If $A$ is a real normed algebra having no nonzero topological divisors of zero, then $A$ is isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

The development of the proof below closely follows Kaplansky's line of reasoning except for changes made to avoid the use of algebraic results not established here and instead to take advantage of Theorem 2.1. Mazur's original proof of Theorem 4.1 used algebraic results analogous to some facts of algebraic number theory and a form of Frobenius' theorem not assuming an identity element. Here, however we will adhere as closely as possible to techniques in the spirit of the theory of normed algebras.

The recurrent theme of the nonemptiness of the spectrum of any element of a complex normed algebra will again surface in dealing with algebras not necessarily having an identity. Consequently a suitable definition of the spectrum is required for algebras without identity, in which every element is *a fortiori* singular. In any ring define a binary operation $x \circ y = x + y - xy$. It is easily verified that $\circ$ is associative and has 0 as a two-sided identity. An element $y$, necessarily unique, is called the *quasi-inverse* of $x$ if $x \circ y = 0 = y \circ x$. If $y$ exists, $x$ is said to be *quasi-regular*, and its quasi-inverse is denoted $x'$. If no such $y$ exists, $x$ is *quasi-singular*. The set of quasi-regular elements of a ring $A$ is then a group under the circle operation. If $A$ has an identity element 1, the relation $1 - (x \circ y) = (1-x)(1-y)$ shows that $x$ is quasi-regular if and only if $1 - x$ is invertible. Guided by these observations, it is standard to formulate a definition of spectrum without reference to an identity. If $x \in A$, an algebra over $\mathbf{C}$, a nonzero complex number $\lambda$ belongs to the *spectrum* $\sigma(x)$ of $x$ if and only if $x/\lambda$ is quasi-singular. The spectrum will contain 0 unless $A$ has an identity and $x$ is invertible. For real algebras the spectrum is defined as before via the complexification, whence a nonzero complex number $\alpha + i\beta \in \sigma(x)$ if and only if $(2\alpha x - x^2)/(\alpha^2 + \beta^2)$ is quasi-singular. It is easy to check that this definition coincides with the original one when $A$ has an identity.

We now survey some basic properties pertaining to the concepts just introduced. In what follows $A$ denotes a complete normed algebra over $\mathbf{R}$ or $\mathbf{C}$ unless the contrary is stated.

PROPOSITION 4.3. *Every $x$ in $A$ satisfying $\|x\| < 1$ is quasi-regular with $x' = -\sum_{n=1}^{\infty} x^n$ and $\|x'\| \leqslant \|x\|/(1-\|x\|)$.*

*Proof.* The geometric series $\sum_{n=1}^{\infty} x^n$ converges by completeness and $x - \sum_{n=1}^{\infty} x^n + x \sum_{n=1}^{\infty} x^n = 0$. The bound on $\|x'\|$ follows by applying the triangle inequality to $x' = -x + xx'$.

PROPOSITION 4.4. *If $y$ is quasi-regular, so is $y + x$ for $\|x\| < k = 1/(1+\|y'\|)$, and $\|(y+x)' - y'\| \leqslant \|x\|/(k-\|x\|)\,k$.*

*Proof.* If $\| x \| < k$, $\| x - xy' \| \leqslant \| x \| (1+\|y'\|) < 1$, so $u = x - xy'$ is quasi-regular. Since $(y+x) \circ y' = u$, $y' \circ u'$ is a right quasi-inverse for $y + x$. Repeating the argument for $x - y'x$ we find that $y + x$ also has a left quasi-inverse. Thus $y + x$ is quasi-regular and $(y+x)' = y' \circ u'$. Moreover $(y+x)' - y' = y' \circ u' - y' = u' - y'u'$, so $\| (y+x)' - y' \| \leqslant (1+\|y'\|) \| u' \| \leqslant (1+\|y'\|) \| u \|/(1-\|u\|) \leqslant \| x \| (1+\|y'\|)^2/ (1-\|x\| (1+\|y'\|)) = \| x \|/(k-\|x\|) k$.

COROLLARY 4.5. (a) *The set* $QR$ *of quasi-regular elements of* $A$ *is open and* $y \to y'$ *is continuous.*

(b) *If* $A$ *has an identity, the set* $G$ *of invertible elements is open and* $y \to y^{-1}$ *is continuous.*

*Proof.* Part (a) follows immediately from Propositions 4.3 and 4.4. Since $G = 1 - QR$ and $y^{-1} = 1 - (1-y)'$ for $y \in G$, (b) follows from (a).

COROLLARY 4.6. *The spectrum of any element is closed and bounded.*

*Proof.* If $\lambda \in \sigma(x)$, $\lambda \neq 0$, $x/\lambda$ is quasi-singular so $\| x/\lambda \| = | \lambda |^{-1} \|x\| \geqslant 1$ by Proposition 4.3. Thus $|\lambda| \leqslant \|x\|$ for all $\lambda \in \sigma(x)$. If $\lambda \notin \sigma(x)$, $\lambda \neq 0$, then $x/\lambda$ is quasi-regular. Since $QR$ is open $x/\mu$ will be quasi-regular for all $\mu$ sufficiently near $\lambda$. If $0 \notin \sigma(x)$, then $x \in G$, so $\lambda - x \in G$ for all $\lambda$ sufficiently near 0. Thus $\mathbf{C} \setminus \sigma(x)$ is open and $\sigma(x)$ is closed.

COROLLARY 4.7. *If* $y$ *is quasi-regular, then* $[(y+\lambda y)' - y']/\lambda \to (y')^2 - y'$ *as* $\lambda \to 0$.

*Proof.* Taking $x = \lambda y$ in Proposition 4.4 so that $u = \lambda(y - yy') = -\lambda y'$ we have $[(y+\lambda y)' - y']/\lambda = [(-\lambda y')' - y'(-\lambda y')']/\lambda = \sum_{n=1}^{\infty} (-1)^n \lambda^{n-1} [(y')^n - (y')^{n+1}] \to (y')^2 - y'$ as $\lambda \to 0$.

Even if $A$ has no identity element we may speak of $\lambda - x$ being a topological divisor of zero for any scalar $\lambda$, i.e., $\lambda y_n - xy_n \to 0$ or $\lambda y_n - y_n x \to 0$ for a sequence $y_n$, $\| y_n \| = 1$. With this convention we have the following result:

PROPOSITION 4.8. (a) *If* $x$ *belongs to the frontier of* $QR$, *then* $1 - x$ *is a t.d.z.*

(b) *If* $A$ *has an identity and* $x$ *belongs to the frontier of* $G$, *then* $x$ *is a t.d.z.*

*Proof.* (a) Since $QR$ is open, $x$ is quasi-singular and $x_n \to x$ where $x_n \in QR$. Now $\{x_n'\}$ is not bounded; for otherwise $x \circ x_n' = (x - x_n) - (x - x_n) x_n'$ implies that $\| x \circ x_n' \| < 1$ and hence $x \circ x_n' \in QR$ for large $n$. This would mean $x \in QR$. Let $z_n = x_n' / \| x_n' \|$. Then $z_n - xz_n = (x - x_n) / \| x_n' \| - (x - x_r) z_r - x / \| x_n' \| \to 0$. Part (b) follows from (a) since $G = 1 - QR$.

PROPOSITION 4.9. *Suppose $A$ is a real normed algebra. If $x$ is a t.d.z. in the completion or complexification of $A$, then $x$ is a t.d.z. in $A$.*

*Proof.* In the case of the complexification we suppose $x$ is a t.d.z. in $A_{\mathrm{C}}$, so we have, say, $x(a_n + ib_n) \to 0$ where $a_n, b_n \in A$ and $\| a_n + ib_n \| = 1$. Then $xa_r \to 0$ as $xb_r \to 0$. Not both $\{a_n\}$ and $\{b_n\}$ can converge to zero, for then $a_n + ib_n \to 0$. If $a_n \to 0$, by taking subsequences if necessary and normalizing we can obtain $xa_n \to 0$ where $\| a_n \| = 1$.

In the case of the completion suppose $x$ is a t.d.z. in the completion $A'$ of $A$. Then $xy_n \to 0$ for $y_n \in A'$ and $\| y_n \| = 1$. Using the density of $A$ in $A'$ and constructing a suitable "diagonal" sequence we can choose $z_n \in A$ such that $xz_r \to 0$ and $z_n \to 0$. It follows that $x$ is a t.d.z. in $A$.

The following lemma, crucial to the proof of Theorem 4.2, employs the same technique Gelfand used to prove the Mazur-Gelfand theorem to establish a necessary condition for the spectrum of an element to be "degenerate", i.e., $\{0\}$.

LEMMA 4.10. *If $\sigma(x) = \{0\}$, then $x$ is a t.d.z.*

*Proof.* We may, of course, assume that $x \neq 0$, and by Proposition 4.9 it suffices to consider the complex case. Since $\lambda x \in QR$ for all $\lambda \in \mathbf{C}$ we can consider $\{(\lambda x)' : \lambda \in \mathbf{C}\}$. This set must be unbounded. Otherwise for any continuous linear functional $f$ on $A$, $\lambda \to f\ [(\lambda x)']$ would be bounded and by Corollary 4.7 entire. Liouville's theorem implies that $f$ must be constant. Therefore $f(0) = f(1)$, i.e., $0 = f(x')$. Since $f$ was arbitrary the Hahn-Banach theorem implies that $x' = 0$, so $x = 0$, contradicting our assumption. Thus we may choose a complex sequence $\lambda_n \to \infty$ such that $y_n = (\lambda_n x)'$ is unbounded. Letting $z_r = y_n / \| y_n \|$ we have $xz_n = y_n x y_n / \lambda_n \| y_n \| = x / \| y_n \| + z_n / \lambda_n \to 0$. Thus $x$ is a t.d.z.

By examining the proof of Corollary 4.7 in the case where it is applied above and by using the "completion" part of Proposition 4.9 one can omit the completeness of $A$ from the hypothesis of Lemma 4.10. We shall, however, not need this in the proof of Theorem 4.2 to which we now turn.

*Proof of Kaplansky's Theorem* (4.2). The crux of the proof is to show that $A$ must have an identity element. Then it is easy to see that $A$ is a division algebra so that Mazur's Theorem 2.1 applies. Let $x$ be any nonzero element of $A$ and let $B$ be the complexification of the completion of $A$. Since $x$ is not a t.d.z. in $B$, $\sigma(x)$ contains nonzero elements; it is also closed and bounded. Therefore we may choose $\lambda = \alpha + i\beta \neq 0$ belonging to the frontier of $\sigma(x)$. Then $z = (2x - x^2)/(\alpha^2 + \beta^2)$ is a quasi-singular element which is the limit of quasi-regular elements. By Proposition 4.8 $1 - z$ is a t.d.z. in $B$, hence in $A$. For each $y$ in $A$ it follows that $y - yz$ and $y - zy$ are t.d.z.; thus $yz = y = zy$ for all $y$ in $A$. So $A$ has an identity element. The set $S = \{x \in A : x \neq 0 \text{ and } x \text{ is singular}\}$ is open since $S$ does not meet the frontier of $G$. Then $A \setminus \{0\} = G \cup S$, where $G \cap S = \varnothing$, so either $A \setminus \{0\}$ is disconnected or $S = \varnothing$. The former condition occurs only if $\dim_{\mathbf{R}} A = 1$ [see Lemma 4.12 below for details.] Then $A \simeq \mathbf{R}$ and the conclusion holds. The latter immediately implies that $A$ is a division algebra, and Theorem 2.1 concludes the proof.

Since a normed algebra satisfying $\|xy\| = \|x\| \, \|y\|$ can have no non-zero topological divisors of zero, Theorem 4.2 implies Theorem 4.1. The hypothesis of 4.1 can be weakened slightly:

COROLLARY 4.11. *If $A$ is a real normed algebra satisfying $\|xy\| \geqslant \beta \|x\| \, \|y\|$ for some positive constant $\beta$ and all $x$, $y$, then $A$ is isomorphic to* **R**, **C**, *or* **H**.

If the multiplicative norm condition $\|xy\| = \|x\| \, \|y\|$ holds for all $x$, $y$ in a Banach algebra $A$ with unit $e$ and $\|e\| = 1$, then certainly $1 = \|xx^{-1}\| = \|x\| \, \|x^{-1}\|$, i.e., $\|x^{-1}\| = \|x\|^{-1}$ for all invertible $x$ in $A$. R. E. Edwards [37] has shown that in a real Banach algebra with identity this condition also characterizes the classical division algebras over **R**. This follows in an elementary way from Mazur's Theorem 2.1. We begin with two lemmas both of which are well-known.

LEMMA 4.11. *Let $A$ be a Banach algebra with identity $e$ and $G$ the group of invertible elements of $A$. If $\{x_r\} \subset G$, $\|x_n^{-1}\|$ is a bounded sequence, and $x_n \to x$, then $x \in G$.*

*Proof.* Suppose $\|x_n^{-1}\| \leqslant M$ for every $n$. The identity $x_n^{-1} - x_m^{-1} = x_n^{-1}(x_m - x_n)x_m^{-1}$ then implies that $\|x_n^{-1} - x_m^{-1}\| \leqslant M^2 \|x_n - x_m\|$; hence $\{x_n^{-1}\}$ is Cauchy. Let $y = \lim x_n^{-1}$. The continuity of multiplication gives $xy = e = yx$ so that $x \in G$.

LEMMA 4.12. *If $A$ is a normed linear space with $\dim_{\mathbf{R}} A \geqslant 2$, then $A \setminus \{0\}$ is connected.*

*Proof.* Let $a$ and $b$ belong to $A \setminus \{0\}$. If $a$ and $b$ are independent over $\mathbf{R}$, the segment $\{ta + (1-t)b : 0 \leqslant t \leqslant 1\}$ does not contain 0. If $a$ and $b$ are dependent over $\mathbf{R}$, we can choose $c$ independent of $a$ (hence of $b$) over $\mathbf{R}$ since $\dim_{\mathbf{R}} A \geqslant 2$. Thus both $a$ and $b$ can be joined to $c$ by a line not containing 0. This proves $A \setminus \{0\}$ is arcwise connected.

THEOREM 4.13. [Edwards]. *If $A$ is a Banach algebra with identity such that $\| x^{-1} \| = \| x \|^{-1}$ for all invertible $x$, then $A$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. If $A$ is a complex Banach algebra satisfying this condition, $A$ is isomorphic to $\mathbf{C}$.*

*Proof.* If $\dim_{\mathbf{R}} A = 1$, the conclusion is obvious. Otherwise we show that $A$ is a division algebra and apply Mazur's theorem. According to Corollary 4.5 (b), the set $G$ of invertible elements is open in $A$, hence in $A \setminus \{0\}$. We show that $G$ is also closed in $A \setminus \{0\}$. Let $x_n \in G$ and $x_n - x \neq 0$. Then $\| x_n^{-1} \| = \| x_n \|^{-1}$. Since $\| x_n \|^{-1} \to \| x \|^{-1} > 0$, we see that $\| x_n \|^{-1}$ is a bounded sequence of real numbers. By Lemma 4.11, $x \in G$ with $x_n^{-1} \to x^{-1}$. As we are assuming $\dim_{\mathbf{R}} A \geqslant 2$, $A \setminus \{0\}$ is connected by Lemma 4.12 and $G = A \setminus \{0\}$ as required.

COROLLARY 4.14. *If $A$ is a Banach algebra with identity and there is a constant $\alpha$ satisfying $\| x \| \, \| x^{-1} \| \leqslant \alpha \| xx^{-1} \|$ for all invertible $x$, then $A$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.*

*Proof.* This weaker condition on the norm also guarantees that $\| x_n^{-1} \|$ is a bounded sequence.

M. Seetharama Gowda [42] has given a different proof of the complex version of Edwards' theorem based on numerical range. He requires only that the set $\{x \in A : \| x \| \, \| x^{-1} \| = 1\}$ have nonempty interior. This apparently weaker condition is also proved sufficient by S. Aurora [17] in a version of Edwards' theorem in the more general context of metric rings.

In the direction of the results of Gowda and Aurora one might study the impact of imposing various conditions dealt with in this survey on a nonempty open subset of the algebra rather than on the whole algebra. For example, $\| xy \| = \| x \| \, \| y \|$ where $x$, $y$ range over an open set $U$, or $\| xx' \| = \| x \| \, \| x' \|$ for all $x$ in an open subset $U$ of $QR$. These conditions can be considered for algebras with or without identity. As we shall see later B. Aupetit [9] has obtained some theorems of this kind.

By considering the incomplete normed algebra of polynomials with complex coefficients and norm $\| \sum_{i=0}^{n} c_i x^i \| = \sum_{i=0}^{n} | c_i |$, it is easy to see that completeness is essential in Edwards' theorem.

## 5. Norm conditions and commutativity

The investigation of further conditions on the norm of a normed algebra lead to consideration of the *spectral radius*, defined by

$$\rho(x) = \lim_{n \to \infty} \| x^n \|^{1/n} .$$

We list several of its properties for later reference (see [73, pp. 10, 30]). For elements $x$ and $y$ in a normed algebra $A$:

(1) $\rho(x) \leqslant \| x \|$.

(2) $\rho(xy) = \rho(yx)$ and $\rho(x^n) = \rho(x)^n$.

(3) $\rho(x+y) \leqslant \rho(x) + \rho(y)$ and $\rho(xy) \leqslant \rho(x)\rho(y)$ if $xy = yx$.

(4) $\rho(x) = \sup \{ | \lambda | : \lambda \in \sigma(x) \}$ provided $A$ is complete.

In view of the exceptionally strong consequences of the multiplicative norm condition $\| xy \| = \| x \| \, \| y \|$ it is natural to inquire into the algebraic implications of the similar condition

$$\| x^2 \| = \| x \|^2 . \tag{*}$$

One familiar consequence of (*) in a normed algebra $A$ is that

$$\rho(x) = \| x \| \tag{**}$$

for all $x$ in $A$. On the other hand since $\rho$ always satisfies $\rho(x^2) = \rho(x)^2$, the conditions (*) and (**) are equivalent in any normed algebra. Their importance can be surmised by noting that they imply the Gelfand representation for commutative Banach algebras is isometric.

Almost simultaneously Claude Le Page [54] and R. A. Hirschfeld together with W. Żelazko [47] discovered independently that (*) in fact implies the commutativity of $A$. Although both papers use essentially the same techniques, the exponential function and Liouville's theorem, they are complementary in the direction of corollaries pursued. We shall survey these two papers here providing some simplifications of proofs and generalizations of the theorems when possible.

THEOREM 5.1. [Le Page, Hirschfeld-Żelazko]. *If $A$ is a normed algebra over* **C** *satisfying $\| x \|^2 \leqslant \alpha \| x^2 \|$ for all $x$ in $A$ and some constant $\alpha$, then $A$ is commutative.*

*Proof.* The inequality $\| x \|^2 \leqslant \alpha \| x^2 \|$ extends to the completion of $A$ so that we may assume $A$ is a Banach algebra. Iterating the given estimate, $\| x \|^{2n} \leqslant \alpha^{2n-1} \| x^{2n} \|$ or $\| x \| \leqslant \alpha^{1-1/2n} \| x^{2n} \|^{1/2n} \to \alpha \rho (x)$ as $n \to \infty$.

If $A$ has no identity, adjoin one to obtain $A_1$, which contains $A$ isometrically as a closed two-sided ideal. (We do not claim that $\| x \| \leqslant \alpha \rho (x)$ holds on all of $A_1$.) Let $x$ and $y$ be arbitrary elements of $A$, and set $z(\lambda) = \exp (\lambda x) \, y \exp (-\lambda x)$, $\lambda \in C$. If $L$ is any continuous linear functional on $A$, define $f(\lambda) = L(z(\lambda))$. Then $f$ is an entire function of $\lambda$. Since $\exp(-\lambda x) = \exp(\lambda x)^{-1}$ (in $A_1$ if $A$ has no identity element) and $z(\lambda) \in A$, $\| z(\lambda) \| \leqslant \alpha \rho (\exp (\lambda x) \, y \exp (-\lambda x)($ $= \alpha \rho (y)$. Therefore $f$ is bounded and entire. By Liouville's Theorem $f$ is constant so $f(\lambda) = L(y) = L(z(0))$. Differentiating $f$ with respect to $\lambda$ by the product rule and setting $\lambda = 0$, we have $0 = f'(0) = L(\exp(\lambda x)(xy - yx) \exp(-\lambda x)) |_{\lambda = 0} = L(xy - yx)$. By the Hahn-Banach Theorem, $xy - yx = 0$ since $L$ was arbitrary.

As a corollary we obtain a relation between the spectral radius and the norm which implies commutativity. This is the form which the principal theorem of Hirschfeld-Żelazko takes.

COROLLARY 5.2. [Hirschfeld-Żelazko]. *If $A$ is a complex normed algebra satisfying $\| x \| \leqslant \alpha \rho (x)$ for all $x$ in $A$ and some constant $\alpha$, then $A$ is commutative.*

*Proof.* Squaring the relation $\| x \| \leqslant \alpha \rho (x)$, we have $\| x \|^2 \leqslant \alpha^2 \rho (x)^2 = \alpha^2 \rho (x^2) \leqslant \alpha^2 \| x^2 \|$; so the hypothesis of Theorem 5.1 holds with the constant $\alpha^2$.

Corollaries 5.3-5.7 below comprise the remaining results of the Hirschfeld-Żelazko paper and follow fairly easily from Corollary 5.2. An element of a normed algebra is called *quasi-nilpotent* if its spectral radius is zero.

COROLLARY 5.3. *If $A$ is a complex normed algebra in which $0$ is the only quasi-nilpotent element and $\rho$ is subadditive and submultiplicative, then $A$ is commutative.*

*Proof.* Under the hypotheses $\rho$ is a norm on $A$. Apply Theorem 5.1 to the normed algebra $(A, \rho)$ with $\alpha = 1$.

In Corollary 5.3 if $A$ is complete we can omit the hypothesis that $\rho$ is submultiplicative, and follow the *proof* of Theorem 5.1: Take $L$ to be any $\rho$-continuous (hence $\|\cdot\|$-continuous) linear functional on $A$ and observe that $z(\lambda)$ is $\rho$-bounded. Applying the Hahn-Banach Theorem to the normed linear space $(A, \rho)$ we obtain $xy = yx$ as before.

COROLLARY 5.4. *If $A$ is a complex Banach algebra and $\rho$ is subadditive and submultiplicative, then $\rho(xy - yx) = 0$ for all $x$, $y$ in $A$.*

*Proof.* Since $\rho(xy) \leqslant \rho(x)\rho(y)$, $N = \{x \in A : \rho(x) = 0\}$ is an ideal of $A$. $N$ is closed because $\rho$ is continuous, being subadditive and dominated by $\|\cdot\|$. Thus $A/N$ is a Banach algebra. The spectral radius on $A/N$ satisfies $\rho(x + N) = \rho(x)$ for every $x$ in $A$ since $\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$ and each element of $N$ is quasi-regular [See Lemma 6.1 below.] Thus $\rho$ on $A/N$ is likewise subadditive and submultiplicative. Corollary 5.3 implies that $A/N$ is commutative so that $xy - yx \in N$ as required.

In order to state the next corollary we need to define the (Jacobson) radical of an algebra, which will also play a prominent role in the latter part of this paper. A left ideal $I$ of an algebra $A$ is called *modular* if there is an element $u$ in $A$ satisfying $xu - x \in I$ for all $x$ in $A$. The *radical* of $A$, denoted $\mathrm{Rad}(A)$ is the intersection of all maximal modular left ideals of $A$. Some relevant facts about the radical at this point are that $\mathrm{Rad}(A) \subset N = \{x \in A : \rho(x) = 0\}$ and $\mathrm{Rad}(A)$ is the largest two-sided ideal of $A$ in which every element is quasi-regular (See Rickart [73, pp. 55-57].)

COROLLARY 5.5. *If $A$ is a complex Banach algebra and $\rho$ is subadditive and submultiplicative, then $A/\mathrm{Rad}(A)$ is commutative.*

*Proof.* Since $N$ is an ideal under the given conditions, the properties of the radical mentioned above imply that $N = \mathrm{Rad}\, A$. An application of Corollary 5.4 yields the desired result.

We shall see subsequently that either the subadditivity or the submultiplicativity of $\rho$ separately imply that $A/\mathrm{Rad}(A)$ is commutative.

COROLLARY 5.6. *In a non-commutative complex normed algebra,* $\inf\limits_{x \neq 0} \rho(x/\|x\|) = 0$ *and* $\inf\limits_{x \neq 0} \|x^2\|/\|x\|^2 = 0.$

*Proof.* If either infinum is positive, an inequality of the kind $\| x \| \leqslant \alpha\rho(x)$ or $\| x \|^2 \leqslant \alpha \| x^2 \|$ would hold, implying commutativity.

COROLLARY 5.7. *Every non-commutative finite dimensional normed algebra contains a non-zero nilpotent element.*

*Proof.* Since the unit sphere is compact the continuous function $x \to \| x^2 \|$ assumes the value of its infimum.

Le Page's study of commutativity considers conditions on the norm directly rather than on the spectral radius.

THEOREM 5.8. [Le Page]. *If $A$ is a complex normed algebra with identity such that $\| xy \| \leqslant \alpha \| yx \|$ for all $x$, $y$ in $A$ and some constant $\alpha$, then $A$ is commutative.*

*Proof.* The norm condition extends to the completion of $A$ so we may assume that $A$ is a Banach algebra. Let $z(\lambda) = \exp(\lambda x)\, y \exp(-\lambda x)$ for $\lambda$ in $\mathbf{C}$. Then $\| z(\lambda) \| \leqslant \alpha \| y \|$ so the proof is completed as in Theorem 5.1.

This theorem has been improved by Baker and Pym [27] to require that $A$ have only a bounded approximate identity. Their method uses the exponential function and Liouville's Theorem in much the same way as Le Page. We do not know if 5.8 holds without an identity assumption.

THEOREM 5.9. [Le Page]. *If $A$ is a normed algebra with identity and $a \in A$ satisfies $\| (a+\lambda)\, x \| \leqslant \| x\, (a+\lambda) \|$ for every $x$ in $A$ and $\lambda$ in $\mathbf{C}$, then $a$ lies in the center of $A$.*

*Proof.* The inequality extends to the completion so we may again assume $A$ is complete. For any $\lambda$ satisfying $| \lambda | > \| a \|$, put $x = y\,(a+\lambda)^{-1}$. Then $(a+\lambda)\, x = (a+\lambda)\, y\, (a+\lambda)^{-1} = (a/\lambda+1)\, y\, (a/\lambda+1)^{-1}$ so $\| (1+a/\lambda)\, y\, (1+a/\lambda)^{-1} \| \leqslant \| y \|$. Thus if $\mu$ is any complex number and $n$ is an integer such that $n > \| a \| \cdot | \mu |$, then $\| (1+\mu/n)\, y\, (1+\mu/n)^{-1} \| \leqslant \| y \|$. Iterating this estimate $n$ times, $\| (1+\mu/n)^n\, y\, (1+\mu/n)^{-n} \| \leqslant \| y \|$, so when $n \to \infty$ we have $\| \exp(\mu a)\, y \exp(-\mu a) \| \; \| y \|$ for all $\mu$ in $\mathbf{C}$ and $y$ in $A$. As in Theorem 5.8, this shows that $ay = ya$ for all $y$ in $A$ as desired.

The radical of a Banach algebra has other characterizations which are needed for the next theorem. A *representation* $\pi$ of a Banach algebra $A$ is a homomorphism from $A$ into $B(X)$, the Banach algebra of bounded linear operators on a Banach space $X$. The representation $\pi$ is called *irre-*

*ducible* provided that $\pi(x)$ has no non-trivial invariant subspaces for every $x$ in $A$. It is *faithful* if $\ker \pi = (0)$. The kernel of an irreducible representation $\pi$ is called a *primitive* ideal of $A$ and has the following intrinsic characterization: a two-sided ideal $J$ is primitive if and only if there is a maximal modular left ideal $L$ such that $J = \{a \in A : ax \in L$ for every $x \in A\}$. The radical of $A$ can then be characterized as the intersection of all primitive ideals of $A$ (See Rickart [73, pp. 54-55]). An algebra $A$ is called *primitive* if (0) is a primitive ideal, *semi-simple* if $\operatorname{Rad}(A) = (0)$, and *radical* if $\operatorname{Rad}(A) = A$.

THEOREM 5.10. [Le Page]. *Suppose $A$ is a semi-simple Banach algebra with identity and set $D_x y = xy - yx$. If one of the two following conditions holds, then $A$ is commutative:*

(1) *For all $x$, $y$ in $A$ and $\varepsilon > 0$ there exists $M > 0$ such that $\| \exp(\lambda x)\, y \exp(-\lambda x) \| \leqslant M \exp(\varepsilon |\lambda|)$ for all $\lambda$ in* $\mathbf{C}$.

(2) *For all $x$, $y$ in $A$, $\| D_x^n y \|^{1/n} \to 0$ as $n \to \infty$.*

*Proof.* Conditions (1) and (2) are equivalent since $\exp(\lambda x)\, y \exp(-\lambda x) = \sum_{n=0}^{\infty} (\lambda^n/n)\, D_x^n y$ (See, for example Markushevich [56, vol. II, p. 259].) If $A$ satisfies (1) or (2), so does every primitive quotient algebra $B = A/J$. But $B$ can be represented faithfully as an irreducible algebra of bounded operators on a Banach space $X$. We wish to show that $B$ is one-dimensional. Suppose, to the contrary, there is an operator $U$ in $B$ and an $a$ in $X$ such that $b = U(a)$ is not a scalar multiple of $a$. Set $D_T U = TU - UT$ and choose $T$ in $B$ such that $T(a) = 0$ and $T(b) = b$, which is possible by the strict transitivity of $B$ (See Rickart [72, p. 60]).

Then $D_T U(a) = TU(a) - UT(a) = b$ and $D_T^n U(a) = b$ for all $n > 0$. Thus $\| D_T^n U \| \geqslant \| b \| / \| a \|$, which contradicts (2). We conclude that every irreducible representation of $A$ is one dimensional, hence commutative. Then $\pi(xy - yx) = 0$ for every $x$, $y$ in $A$ and irreducible representation $\pi$. Since $A$ is semi-simple, $xy - yx = 0$.

Donald Spicer [82] has observed that if every irreducible representation of $A$ is one-dimensional and there exists $k > 0$ such that $\| x \| \leqslant k\rho(x)$ for all $x$ in the commutator $C$ of $A$, then $A$ is commutative; and conversely. Indeed, since every irreducible representation of $A/R$ is one-dimensional, then $A/R$ is commutative and hence $C \subset \operatorname{Rad}(A)$. Therefore every commutator is quasi-nilpotent and it follows that $xy - yx = 0$ by the inequality.

THEOREM 5.11. [Le Page]. *If* $A$ *is a complex Banach algebra with identity and* $Ax^2 = Ax$ *for every* $x$ *in* $A$, *then* $A$ *is commutative and semi-simple.*

*Proof.* To prove that $A$ is semi-simple it suffices to show that 0 is the only quasi-nilpotent element of $A$. If $x$ is quasi-nilpotent, there is a $y$ in $A$ such that $yx^2 = x$. So for every positive integer $p$, $y^p x^{p+1} = x$ or $\|x\|^{1/p} \leqslant \|y\| \; \|x\|^{1/p} \|x^p\|^{1/p} \to 0$ as $p \to \infty$. Thus $\|x\| = 0$ and $x = 0$. Next note that if $yx^2 = x$, $yx$ is idempotent. Since $[x(yx-1)]^2 = x(yx^2-x)(yx-1) = 0$, $x(yx-1)$ is nilpotent, hence zero. Premultiplying by $y$ yields $(yx)^2 - yx = 0$. Now every idempotent $e$ in $A$ is central: for any $z$ in $A$, $ez(1-e)$ and $(1-e)ze$ are nilpotent, hence zero. Thus $ez$ and $ze$ are both $eze$.

Now the hypothesis of 5.11 will be fulfilled in every quotient and in particular every primitive quotient $B = A/J$ of $A$. If $x$ is a non-zero element of $B$, there is a $y$ in $B$ such that $yx$ is nonzero and central. Since a central element in an algebra of operators is multiplication by a scalar, $x$ is left invertible. This implies $B \simeq \mathbf{C}$ and so every irreducible representation of $A$ is one-dimensional. As in Theorem 5.10 we conclude that $A$ is commutative.

J. Duncan and A. Tullo [36] have extended Theorem 5.11 to show that $A$ must be finite-dimensional and in fact $A \approx \mathbf{C}^n$.

It is easily seen that the extended result of Duncan and Tullo fails if $A$ is not required to have identity; e.g., let $A$ be the space of complex sequences which are eventually zero.

THEOREM 5.12. [Le Page]. *Suppose* $A$ *is a Banach algebra with identity and* $ax - xa$ *is quasi-nilpotent for every* $x$ *in* $A$. *Then* $ax - xa \in \operatorname{Rad}(A)$ *for all* $x$ *in* $A$.

*Proof.* We proceed as in the proof of Theorem 5.10, first noting that the hypothesis on a holds for its canonical image in any quotient of $A$. Let $B = A/J$ be any primitive quotient, considering $B$ as an irreducible algebra of operators on $X$, and let $U$ be the image of a in $B$. We show $U$ is in the center of $B$ by showing that it is multiplication by a scalar. This is immediate if $\dim X = 1$, so assume to the contrary that $\dim X > 1$ and that there exists $b \in X$ such that $b' = U(b)$ is not a scalar multiple of $b$. Choose $T$ in $B$ such that $T(b) = 0$ and $T(b') = b$ by the strict transitivity of $B$. Let $V = TU - UT$ so that $V(b) = b$, implying that $1 \in \sigma(V)$ and hence $V$ fails to be quasi-nilpotent. Thus the image of $ax - xa$ is zero by every irreducible representation and consequently belongs to $\operatorname{Rad}(A)$.

We do not know if the assumption of an identity element can be deleted in (5.9)-(5.12). Also, it is open as to whether or not one needs completeness in (5.11) and (5.12).

Hirschfeld and Żelazko closed their influential paper on commutativity [47] with the following two conjectures:

*Conjecture* 1. If for every commutative subalgebra $B$ of a complex Banach algebra $A$ there is a constant $k$ such that $\rho(x) \geqslant k \|x\|$ for every $x$ in $B$, then $A$ is commutative.

*Conjecture* 2. If $A$ is a complex Banach algebra in which 0 is the only quasi-nilpotent and the spectral radius is continuous, then $A$ is commutative.

B. Aupetit has established partial results in the direction of Conjecture 1 [10] and more recently has published a counterexample to Conjecture 2 [15]. In the description which follows the continuity of $x \to \sigma(x)$ refers to the Hausdorff metric on the compact subsets of **C**. A proof of the next result is given in [10, Theorem 1.1].

THEOREM 5.13. [Aupetit]. *If for every* $a$ *in a complex Banach algebra* $A$ *there is a constant* $k > 0$ *such that* $\rho(x) \geqslant k \|x\|$ *for every* $x$ *in the closed subalgebra generated by* $a$, *then the function* $x \to \sigma(x)$ *is locally uniformly continuous on an open dense subset of* $A$.

Global uniform continuity of the spectrum, or even of the spectral radius, on $A$ would imply commutativity modulo the radical as we shall see in the next section. The proof of Theorem 5.13 and the proofs published by Aupetit on commutativity and the spectral radius conditions have been based on potential theory and the use of subharmonic functions. These techniques render the proofs less computational, but also less elementary and accessible. They have however drawn analytical and topological considerations further into the picture and produced some very appealing theorems. For example in another partial result on Conjecture 1, conditions on the topological properties of the spectrum are used to obtain sufficiency for commutativity [10]:

THEOREM 5.14. *If for every* $x$ *in* $A$ *there is an* $a$ *in* $A$ *and* $k > 0$ *such that* $x$ *lies in the closed subalgebra* $C(a)$ *generated by* $a$, $\sigma(a)$ *has no interior points and a finite number of holes, and* $\rho(y) \geqslant k \|y\|$ *for every* $y \in C(a)$, *then* $A$ *is commutative.*

A surprisingly simple counterexample to Conjecture 2 was published by Aupetit in 1978 [15]. He makes extensive use of results obtained by Ackermans [1] in which the Gelfand representation for a commutative Banach algebra $B$ is lifted to the matrix algebra with entries in $B$. Let $U$ be the open unit disk in $\mathbf{C}$ and $B$ the commutative Banach algebra of continuous functions on $\bar{U} \times \bar{U}$ which are holomorphic in $U \times U$. In the algebra of $2 \times 2$ matrices with entries in $B$ define the norm by

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max \{ \| a \| + \| b \| , \| c \| + \| d \| \} .$$

Let $A$ be the closed noncommutative subalgebra with 1 formed by the matrices

$$m = \begin{pmatrix} f(z_1, z_2) & g(z_1, z_2) \\ (z_1 + z_2) Tg(z_1, z_2) & Tf(z_1, z_2) \end{pmatrix} \text{ where } f, g \in B ,$$

and $T$ is the isometric automorphism of $B$ defined by $Tf(z_1, z_2) = f(z_2, z_1)$. According to Ackermans [1, Th. 3.1, 3.2] the spectrum is continuous on $A$. If $m \in A$ is quasi-nilpotent then again by [1, Th. 2.2] $\{0\} = \bigcup_{\phi \notin \hat{B}} \sigma(\tilde{\phi}(m))$ where $\hat{B}$ is the set of multiplicative linear functionals on $A$ and

$$\tilde{\phi}(m) = \begin{pmatrix} \phi(f(z_1, z_2)) & \phi(g(z_1, z_2)) \\ \phi((z_1 + z_2) Tg(z_1, z_2)) & \phi(Tf(z_1, z_2)) \end{pmatrix}$$

Thus $\tilde{\phi}(m)$ is quasi-nilpotent for each $\phi \in \hat{B}$ and its square is zero by the Cayley-Hamilton Theorem. Since $B$ is semi-simple, $m^2 = 0$. In particular $f(z_1, z_2)^2 + (z_1 + z_2) g(z_1, z_2) g(z_2, z_1) \equiv 0$, which in turn implies that $f(z_1, z_2) \equiv g(z_1, z_2) \equiv 0$. Hence $m = 0$ so there are no nonzero quasi-nilpotents in $A$.

## 6. Commutativity and the spectral radius

We now consider some weaker conditions on the spectral radius which influence the commutativity of a Banach algebra. Two familiar properties of the norm, subadditivity and submultiplicativity, are also satisfied by the spectral radius on *commuting* elements. Does the imposition of these properties on the whole algebra then imply commutativity? Because

$$|x|_\sigma = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

vanishes for every $x$ in the radical, $|\cdot|_\sigma$ certainly satisfies both properties in a noncommutative radical algebra. Consequently commutativity modulo the radical is the most that could be expected. In 1971, Bernard Aupetit [7] announced that in a Banach algebra $A$ each of these conditions separately imply that $A/\mathrm{Rad}(A)$ is commutative and that in fact the following are equivalent:

(1) $|x + y|_\sigma \leqslant \alpha(|x|_\sigma + |y|_\sigma)$ for some $\alpha > 0$.
("$|\cdot|_\sigma$ is subadditive")

(2) $|xy|_\sigma \leqslant \beta |x|_\sigma |y|_\sigma$ for some $\beta > 0$. ("$|\cdot|_\sigma$ is submultiplicative")

(3) $A/\mathrm{Rad}(A)$ is commutative ($A$ is "almost commutative".)

Proofs of these equivalences were published later [9] using subharmonic functions. Several additional equivalent conditions involving the spectral radius and spectral diameter were added at this time, the most significant of which is

(4) $|\cdot|_\sigma$ is uniformly continuous on $A$.

In the case of algebras with identity Aupetit was in fact able to restrict the subadditive and submultiplicative conditions to a neighborhood of the identity.

Using a more elementary algebraic approach J. Zemánek has also established the equivalence of conditions (1) - (4). An account of this first appeared in a joint paper with V. Pták [70] where three auxiliary conditions were introduced, each equivalent to (1) - (4). A more refined and comprehensive treatment has now been given by Zemánek [112]. This version rests on an analysis of the pseudonorm

$$s(x) = \sup\{|x - u^{-1}xu|_\sigma : u \text{ is invertible in } A \text{ or } A_1\},$$

and depends on an extension of the Jacobson density theorem given in A. M. Sinclair's monograph [79, p. 36]. In [112] Zemánek also established related results involving the spectral radius and used the techniques developed there to prove analogous theorems for real Banach algebras. His work renders the equivalences (1) - (4) accessible without reference to potential theory and subharmonic functions while sacrificing some of the sharpness of Aupetit's results. We now give proofs of these equivalences following the arguments of Pták-Zemánek [70], but supressing the auxiliary conditions mentioned above.

First we note that (3) implies both (1) and (2). This follows immediately from the behavior of $|\cdot|_\sigma$ on the quotient $A/R$, where $R = \text{Rad}(A)$.

LEMMA 6.1. *For any $x$ in the Banach algebra $A$, $|x + R|_\sigma = |x|_\sigma$.*

*Proof.* We show that $\sigma(x+R) \cup \{0\} = \sigma(x) \cup \{0\}$. Let $\lambda \neq 0$ belong to $\sigma(x)$ so that $x/\lambda$ is quasi-singular in $A$. If $x/\lambda + R$ were quasi-regular in $A/R$, then $x/\lambda \circ y \in R$ for some $y$ in $A$. Since every element of $R$ is quasi-regular, $x/\lambda$ would be quasi-regular also. Thus $\lambda \in \sigma(x+R)$. The reverse inclusion is similarly proved and the result clearly extends to any quasi-regular ideal.

PROPOSITION 6.2. *If $A$ is an almost commutative Banach algebra, then $|\cdot|_\sigma$ is subadditive and submultiplicative.*

*Proof.* $|x + y|_\sigma = |x + y + R|_\sigma \leqslant |x + R|_\sigma + |y + R|_\sigma = |x|_\sigma + |y_\sigma|$. Analogously $|xy|_\sigma \leqslant |x|_\sigma \cdot |y|_\sigma$.

PROPOSITION 6.3. *If $|\cdot|_\sigma$ is subadditive, then $|\cdot|_\sigma$ is uniformly continuous.*

*Proof.* The subadditivity of $|\cdot|_\sigma$ yields $||x|_\sigma - |y|_\sigma| \leqslant |x y|_\sigma \leqslant \|x - y\|$, which actually shows Lipschitz continuity with a constant of 1.

PROPOSITION 6.4. *If $|\cdot|_\sigma$ is subadditive, then $|\cdot|_\sigma$ is submultiplicative.*

*Proof.* We may certainly assume $\alpha \geqslant 1$. Let $\beta = 9\alpha^2$. To show that $|xy|_\sigma \leqslant 9\alpha^2 |x|_\sigma |y_\sigma|$ it suffices to choose $\lambda \in \mathbf{C}$ such that $|\lambda| > 9\alpha^2 |x|_\sigma |y|_\sigma$ and show $\lambda - xy$ is invertible. (We now adjoin an identity to $A$ if it has none, but (2) is assumed to hold only in $A$). Choose complex numbers $\mu$ and $v$ satisfying $\mu v = \lambda$, $|\mu| > 3\alpha |x|_\sigma$, $|v| > 3\alpha |y|_\sigma$. Put $u = x/\mu$ and $v = y/\mu$. Then $|u|_\sigma < 1/3\alpha$ and $|v|_\sigma < 1/3\alpha$. Since $u$ commutes with $(1-u)^{-1}$ we have $|(1-u)^{-1} u|_\sigma \leqslant |(1-u)^{-1}|_\sigma |u|_\sigma < [1/(1-1/3\alpha)]\,[1/3\alpha] = 1/3\alpha - 1 \leqslant 1/2\alpha$ since $\alpha > 1$. Similarly $|(1-v)^{-1}v|_\sigma < 1/2\alpha$. Now $\lambda - xy$ is invertible if $1 - uv$ is; but $1 - uv = (1-u)\,[1+(1-u)^{-1} u + v(1-v)^{-1}]\,(1-v)$ where each factor is invertible, the middle one because $|(1-u)^{-1} u + v(1-v)^{-1}|_\sigma \leqslant \alpha |(1-u)^{-1} u|_\sigma + \alpha |v(1-v)^{-1}|_\sigma < 1$.

PROPOSITION 6.5. *If $|\cdot|_\sigma$ is submultiplicative, then $|\cdot|_\sigma$ is subadditive.*

*Proof.* It is convenient to consider the cases with and without identity separately. Suppose $A$ has an identity and $|\lambda| > \beta |x|_\sigma + \beta |y|_\sigma$ (assume

$\beta \geqslant 1$). Then $\lambda - x$ is invertible and $\lambda - (x+y) = (\lambda - x)\,[1-(\lambda-x)^{-1}y]$ while $|(\lambda-x)^{-1}y|_\sigma \leqslant \beta\,|(\lambda-x)^{-1}|_\sigma\,|y|_\sigma \leqslant \beta\,(|\lambda|-|x|_\sigma)^{-1}\,|y|_\sigma < 1$. Thus $\lambda - (x+y)$ is invertible and the conclusion follows with $\alpha = \max\{\beta, 1\}$. If $A$ has no identity, again assume $\beta \geqslant 1$ and that $|\lambda| > \beta\,|x|_\sigma + \beta\,|y|_\sigma$. Since $\lambda - x$ is invertible in $A_1$, we have $(\lambda-x)^{-1} = v + u$ where $v \in A$ and $\mu \in \mathbf{C}$. From $(\lambda-x)\,(\mu+v) = 1$ we have $\mu = 1/\lambda$, and hence $v = (\lambda-x)^{-1} - 1/\lambda$. Now $\lambda - (x+y) = (\lambda-x)\,[1-(\lambda-x)^{-1}y] = (\lambda-x)\;[1-(1/\lambda y - vy] = (\lambda-x)\;[1-vy\,(1-(1/\lambda)y)^{-1}]\;(1-(1/\lambda)y)$, where $1 - (1/\lambda)y$ is invertible since $|\lambda| > |y|_\sigma$. Since $A$ is an ideal in $A_1$, we have $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|v|_\sigma\,|y\,(1-(1/\lambda)y)^{-1}|_\sigma$. But

$$|v|_\sigma \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda|-|x|_\sigma)} \leqslant \frac{|x|_\sigma}{|\lambda|\,(|\lambda|-\beta|x_{.\sigma})}$$

and $|y\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant |\lambda|\cdot|y|_\sigma/(|\lambda|-|y|_\sigma) \leqslant |\lambda|\cdot|y|_\sigma/(|\lambda|-\beta|y|_\sigma)$. Multiplying these estimates we obtain $|vy\,(1-(1/\lambda)y)^{-1}|_\sigma \leqslant \beta\,|x|_\sigma\,|y|_\sigma/(|\lambda|-\beta|x|_\sigma)\,(|\lambda|-\beta|y|_\sigma) < 1$. So again we may take $\alpha = \max\{\beta, 1\}$.

Proposition 6.6. *If* $|\cdot|_\sigma$ *is subadditive on* $A$, *then* $A$ *is almost commutative.*

*Proof.* Since $|\cdot|_\sigma$ must be submultiplicative, Corollary 5.5 (to the results of Hirschfeld-Żelazko) states that $A/\mathrm{Rad}\,(A)$ must be commutative.

It is of interest that the uniform continuity of $|\cdot|_\sigma$ implies its Lipschitz continuity. It is easy to see that the Lipschitz constant can be taken as $1/\varepsilon$ where $\|x - y\| < \varepsilon$ implies $\|x|_\sigma - |y|_\sigma| \leqslant 1$. We have already seen that the subadditivity of $|\cdot|_\sigma$ implies its Lipschitz continuity.

## 7. Further generalizations and related results

During the past forty years the general subject of this paper has received attention from many authors. Our purpose here is to give a brief discussion of some of the relevant literature.

Ramaswami studies in [71] the Mazur-Gelfand theorem under minimal hypothesis. He weakens the associative law and also the triangle inequality; the former in several ways. In all he gives six different sets of sufficient conditions for a "generalized" complex Banach algebra to coincide with the complex field. He treats real Banach algebras in the same spirit.

Elementary proofs of the Mazur-Gelfand theorem which avoid direct appeal to complex function theory (in particular to Liouville's theorem) have been given by Kametani [51], Ono [64], Ramaswami [71], Rickart [72], Stone [84], and Tornheim [93]. Stone's proof is based on completeness of the algebra and a study of the behavior of the sequence $x^n (x^n - r^n)$ as $n \to \infty$, $r \geqslant 0$, where $x$ is an element which is not a complex multiple of the identity. Tornheim [93] observes that if $A$ is a normed field, $y \in A$, $y \notin \mathbf{C}$, then the function $f : \mathbf{C} \to \mathbf{R}$ defined by

$$f(\lambda) = \| 1/(y - \lambda e) \|$$

is continuous, positive and small for large $\lambda$. Hence $f$ takes on a maximum value $M > 0$ on a closed, bounded subset of $\mathbf{C}$. He then gives an elementary argument (that does not depend on completeness) which shows that this is impossible. The arguments of Ono [64] and Rickart [72] were obtained independently and nearly simultaneously; both are based on properties of roots of unity. Rickart's proof appears in his well known book [73].

Various generalizations of the Mazur-Gelfand theorem and Theorem 3.1 to topological algebras with identity have been considered by Allan [5], Arens [6], Aurora [17], Edwards [37], Ramaswami [71], Shafarevich [76], Shilov [77], Stone [84], Turpin [94], and Żelazko [101-108]. For the most part these papers are concerned with either omitting the assumption of a norm altogether or else significantly weakening the properties of the norm. For example, Arens [6] shows that a convex topological linear division algebra over $\mathbf{C}$ with continuous inversion is isomorphic to $\mathbf{C}$. On the other hand, he observes that the algebra of holomorphic functions on an open subset of $\mathbf{C}$ with the usual topology is a non-normable, convex, metrizable algebra which has no nonzero topological divisors of zero but still is not a division algebra.

The extensive studies of Aurora [17-25] relating to the Mazur-Gelfand theorem and the general classification of topological fields are quite significant. Generally speaking, he centers his attention on associative rings with identity which are furnished with a norm satisfying a wide variety of hypotheses. His results are numerous, and, among other things, contain many of the previously discussed results as special cases. For more information the reader may consult the individual papers.

A very readable account of Żelazko's (and others) work up to 1964 can be found in his Yale lecture notes [104] (see also [105]). His more recent work [106, 107, 108] is concerned with extending known properties of topological divisors of zero. For example, in [106] a topological ring is said to have *generalized topological divisors of zero* if there exists at least one pair of subsets $P$, $Q$ of $A$ such that 0 is in the closure of $PQ$, but is not in the closure of $P$ or $Q$. A locally convex topological algebra $A$ over the complex numbers is called *m-convex* if its topology is defined by a family of semi-norms for which $\| xy \| \leqslant \| x \| \; \| y \|$ for all $x$, $y$ in $A$. The main theorem in [106] states that an $m$-convex topological algebra which is not the complex numbers always has generalized topological divisors of zero. In [107] he shows that a *real* $m$-convex algebra must have generalized topological divisors of zero or be homeomorphically isomorphic to either the reals, complexes, or quaternions. In [108] Kaplansky's theorem (4.2) is extended to real $p$-normed algebras. (Roughly, a *p-normed algebra* is an algebra with a complete algebra norm with the usual homogeneity replaced by $\| \lambda x \| = | \lambda |^p \| x \|$, $0 < p \leqslant 1$.)

The subject of absolute-valued algebras has been studied by Albert [2, 3, 4], Gleichgewicht [41], Strzelecki [85, 86, 87], Urbanik [95], Urbanik and Wright [96], and Wright [99]. In these papers it is not assumed that the algebra is associative. In particular, an example provided in [96] shows that Theorem 3.1 fails for non-associative algebras. In this more general context it is proved by Urbanik and Wright [96] that a real, absolute-valued algebra with identity is isomorphic to either the reals, the complexes, the quaternions, or the Cayley-Dickson algebra. Their proof consists in showing that the algebra in question must be algebraic (i.e., the algebra $A(x)$ generated by $x$ is finite dimensional for each $x$); then the theorem follows from one of Albert's theorems (See [3]).

Two papers on real Hilbert algebras deserve mention. A *Hilbert algebra* is a Hilbert space which is also a Banach algebra relative to the inner product norm $\| x \| = (x|x)^{1/2}$. In the real case Ingelstam proved that if such an algebra has an identity of unit norm, it must be the reals, complexes or quaternions. His argument has been considerably simplified by Smiley [81]. In the complex case Hirschfeld [45] considers *Hilbertizable* algebras, i.e., Banach algebras which support a comparable norm which makes the algebra a Hilbert space, but is not necessarily submultiplicative. Using ideas of Ingelstam and Smiley he shows that a complex Hilbertizable algebra which is semi-simple and has an identity of unit (Hilbert) norm must be the complex numbers. He remarks that van Castern using a different line of argument has shown the assumption of semi-simplicity to be superfluous.

Jan van Castern's similar result, also giving necessary and sufficient conditions for a complex algebra which is a normed linear space to be the complex numbers, is as follows [32]: Let $A$ be an algebra over **C** with identity $e$. Suppose that $A$ as a vector space is normed so that $\| e \| = 1$

and the unit ball is smooth at $e$ (i.e., $\{\phi \in (A, \|\cdot\|)' : \|\phi\| = \phi(e) = 1\}$ is a singleton). Then $A = \mathbf{C}e$ if and only if there is a norm $\|\cdot\|_1$ on $A$ for which $\|(x+e)^n\|_1 \leqslant \exp(n\|x\|)$ for all $n \in \mathbf{N}$ and $x \in A$.

A characterization of commutativity in $C^*$-algebras is given by Crabb-Duncan-McGregor [34] again in terms of a norm inequality: A $C^*$-algebra $A$ is commutative if and only if

$$\|x + y\| \leqslant 1 + \|xy\|$$

for all hermitian elements $x, y \in A$ with $\|x\| = \|y\| = 1$. Some generalizations of this can be found in Duncan and Taylor [35].

A version of Edward's theorem (4.13) using the spectral radius in place of the norm has been established by Aupetit. The statement and proof given in [12] contain a gap which will be corrected in Aupetit's forthcoming monograph [16]. The correct statement is: If $A$ is a real Banach algebra with identity containing a nonempty open set $U$ of invertible elements satisfying $\rho(x)\rho(x^{-1}) = 1$, then $A/\mathrm{Rad}\, A$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or $M_2(\mathbf{R})$. If $A$ is complex, $A/\mathrm{Rad}\, A$ is isomorphic to $\mathbf{C}$.

There are in addition two complex versions of this theorem in the context of $*$-algebras:

Let $A$ be a complex Banach $*$-algebra with identity the set of whose hermitian elements contain a nonempty open set $U$ of invertibles such that for every $x \in U$, $\rho(x)\rho(x^{-1}) = 1$. Then $A/\mathrm{Rad}\, A$ is isomorphic to $\mathbf{C}$ with the involution $u \to \bar{u}$ or to $\mathbf{C}^2$ with the involution $(u, v) \to (\bar{u}, \bar{v})$. The other version has precisely the same statement except that $\rho$ is replaced by $\|\cdot\|$.

Aupetit has also obtained results along the lines of the Hirschfeld-Żelazko theorems in the context of $*$-algebras. We cite the following two theorems from [10]:

(1) Let $A$ be a Banach $*$-algebra and $k > 0$ such that for all hermitian $h$ one has $\rho(h) \geqslant k\|h\|$ and $\sigma(h)$ has no interior points and a finite number of holes. Then if $A$ has no nilpotents, $A$ is commutative.

(2) Let $A$ be a Banach $*$-algebra and $k > 0$ such that for every hermitian $h$ one has $\rho(h) \geqslant k\|h\|$ and $\sigma(h)$ has no interior points and a finite number of holes. Then if $A$ has no non-nilpotent quasi-nilpotents, all its irreducible representations are finite dimensional.

A generalization of commutativity called $P$-commutativity has been studied by W. Tiller in [92]. The definition is as follows: For each positive functional $f$ on a complex $*$-algebra $A$, let $I_f = \{x \in A : f(x^*x) = 0\}$

and $P = \cap I_f$, where the intersection is taken over all positive functionals on $A$. The algebra $A$ is called *P-commutative* if $xy - yx \in P$ for all $x$, $y$ in $A$. Tiller establishes the following two theorems relating properties of the spectral radius to $P$-commutativity:

(1) Let $A$ be a Banach $^*$-algebra which is symmetric and $P$-commutative. Then if $x, y \in A$, $\rho(xy) \leqslant \rho(x)\,\rho(y)$ and $\rho(x+y) \leqslant \rho(x) + \rho(y)$.

(2) Let $A$ be a Banach $^*$-algebra with bounded approximate identity. If $\rho(x^*x) \leqslant \rho(x)^2$ for every $x$ in $A$, then $A$ is $P$-commutative.

## BIBLIOGRAPHY


[1] Ackermans, S. T. M. A case of strong spectral continuity. *Nederl. Akad. Wetensch. Proc. Ser. A 71 = Indag. Math. 30* (1968), 455-459. MR 40 #739.

[2] Albert, A. A. A note of correction. *Bull. Amer. Math. Soc. 55* (1949), 1191. MR 11, 76.

[3] —— Absolute-valued real algebras. *Ann. of Math. (2) 48* (1947), 495-501. MR 8, 561.

[4] —— Absolute-valued algebraic algebras. *Bull. Amer. Math. Soc. 55* (1949), 763-768. MR 11, 76.

[5] Allan, G. R. A spectral theory for locally convex algebras. *Proc. London Math. Soc. (3) 15* (1965), 399-421. MR 31 #619.

[6] Arens, R. Linear topological division algebras. *Bull. Amer. Math. Soc. 53* (1947), 623-630. MR 9, 6.

[7] Aupetit, B. Almost commutative Banach algebras. *Notices Amer. Math. Soc. 18* (1971), 191.

[8] —— Continuité du spectre dans les algèbres de Banach avec involution. *Pacific J. Math. 56* (1975), 321-324. MR 51 #11117.

[9] —— Caractérisation spectrale des algèbres de Banach commutatives. *Pacific J. Math. 63* (1976), 23-35. MR 54 #3409.

[10] —— Sur les conjectures de Hirschfeld et Żelazko dans les algèbres de Banach. *Bull. Soc. Math. France 104* (1976), 185-193. MR 54 #8290.

[11] —— On scarcity of operators with finite spectrum. *Bull. Amer. Math. Soc. 82* (1976), 485-486. MR 53 #3699.

[12] —— Caractérisation spectrale des algèbres de Banach de dimension finie. *J. of Funct. Anal. 26* (1977), 232-250. MR 56 #12887.

[13] —— Continuité uniforme du spectre dans les algèbres de Banach avec involution. *C. R. Acad. Sci. Paris, Ser. A-B, 284* (1977), A1125-A1127. MR 55 #8806.

[14] —— Continuité et uniforme continuité du spectre dans les algèbres de Banach. *Studia Math. 61* (1977), 99-114. MR 56 #12883.

[15] —— La deuxième conjecture de Hirschfeld-Żelazko pour les algèbres de Banach est fausse. *Proc. of Amer. Math. Soc. 70* (1978), 161-162.

[16] —— *Propriétés spectrales des algèbres de Banach.* Springer Lecture Notes, to appear.

[17] Aurora, S. Multiplicative norms for metric rings. *Pacific J. Math. 7* (1957), 1279-1304. MR 19, 1186.

[18] —— On power multiplicative norms. *Amer. J. Math. 80* (1958), 879-894. MR 20 #5192.

[19] AURORA, S. The embedding of certain metric fields. *Mich. Math. J. 7* (1960), 123-128. MR 22 #12171.

[20] —— A note on certain connected metric division rings. *Mich. Math. J.* 7 (1960), 129-132. MR 22 #12172.

[21] —— On normed algebras whose norms satisfy polynomial identities. *Canadian J. Math. 13* (1961), 664-674. MR 24 #A2258.

[22] —— On normed algebras which satisfy a reality condition. *Canadian J. Math. 13* (1961), 675-682. MR 24 #A2259.

[23] —— On connected normed algebras whose norms satisfy a reality condition. *Illinois J. Math. 6* (1962), 488-499. MR 26 #620.

[24] —— On normed rings with monotone multiplication. *Pacific J. Math. 33* (1970), 15-20. MR 41 #4235.

[25] —— Normed fields which extend normed rings of integers. *Pacific J. Math. 33* (1970), 21-28. MR 42 #1810.

[26] BAER, R. Normierte Funktionenalgebren. *J. Reine Angew. Math. 204* (1960), 41-66. MR 25 #5397.

[27] BAKER, J. W. and J. S. PYM. A remark on continuous bilinear mappings. *Proc. Edinburgh Math. Soc. (2) 17* (1970/71), 245-248. MR 46 #2429.

[28] BAUMANN, V. Normierte Funktionenalgebren, II. *J. Reine Angew. Math. 204* (1960), 67-73. MR 25 #5398.

[29] BOEHME, T. K. On the limits of the Gelfand-Mazur theorem. *Proc. Conf. on Convergence Spaces*, Univ. of Nevada, Reno, 1976. MR 55 #3788.

[30] BONSALL, F. and J. DUNCAN. *Completed Normed Algebras.* Springer-Verlag, New York, 1973. MR 54 #11013.

[31] BOYADŽIEV, H. Commutativity in Banach algebras and elements with real spectra. *C. R. Acad. Bulgare Sci. 29* (1976), 1401-1403. MR 55 #8802.

[32] VAN CASTEREN, J. On algebras with a smooth norm. *Bull. Soc. Math. Belgique 26* (1974), 347-350. MR 56 #16378.

[33] CHABAUTY, R. Une généralisation du théorème de Gelfand-Mazur. *C. R. Acad. Sci. Paris, Ser. A-B, 275* (1972), A519-A522. MR 46, 2430.

[34] CRABB, M. J., J. DUNCAN and C. M. MC GREGOR. Characterizations of commutativity for $C^*$-algebras. *Glasgow Math. J. 15* (1974), 172-175. MR 50#14252.

[35] DUNCAN, J. and P. J. TAYLOR. Norm inequalities for $C^*$-algebras. *Proc. Roy. Soc. Edinburgh, Sect. A, 75* (1975/76), no. 2, 119-129. MR 56 #12896.

[36] DUNCAN, J. and A. TULLO. Finite dimensionality, nilpotents and quasi-nilpotents in Banach algebras. *Proc. Edinburgh Math. Soc. (2) 19* (1974/75), 45-49. MR 49 #9631.

[37] EDWARDS, R. E. Multiplicative norms on Banach algebras. *Proc. Camb. Philos. Soc. 47* (1951), 473-474. MR 13, 256.

[38] GELFAND, I. M. Normierte Ringe. *Mat. Sbornik 9* (1941), 3-24. MR 3, 51.

[39] GELFAND, I. M., D. A. RAIKOV and G. E. SHILOV. Commutative normed rings. *Uspehi Matem. Nauk. 1* (1946), 48-146. MR 10, 258; *Amer. Math. Soc. Transl. (2) 5* (1957), 115-220. MR 18, 714.

[40] GELFAND, I. M., D. A. RAIKOV and G. E. SHILOV. *Commutative normed rings.* Chelsea, New York, 1964. MR 34 #4940.

[41] GLEICHGEWICHT, B. A remark on absolute valued algebras. *Colloq. Math. 11* (1963), 29-30. MR 28 #5349.

[42] GOWDA, M. S. Numerical range proof of Edwards' theorem. *Tamkang J. Math. 8* (1977), 77-80. MR 56 #12884.

[43] HERSTEIN, I. N. *Topics in algebra.* Blaisdell, (Waltham, Mass.) 1964. MR 30 #2028.

[44] HILLE, E. and R. S. PHILLIPS. *Functional analysis and semigroups.* Amer. Math. Soc., Providence, R. I., 1957. MR 19, 664.

[45] HIRSCHFELD, R. A. On Hilbertizable Banach algebras. *Bull. Soc. Math. Belgique 25* (1973), 331-333. MR 56 #16377.

[46] HIRSCHFELD, R. A. and S. ROLEWICZ. A class of non-commutative Banach algebras without divisors of zero. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 17* (1969), 751-753. MR 40 #7802.

[47] HIRSCHFELD, R. A. and W. ZELAZKO. On spectral norm Banach algebras. *Bull. Acad. Pol. Sci. 16* (1968), 195-199. MR 37 #4621.

[48] INGELSTAM, L. Hilbert algebras with identity. *Bull. Amer. Math. Soc. 69* (1963), 794-796. MR 27 #4096.

[49] —— Non-associative normed algebras and Hurwitz' problem. *Ark. Mat. 5* (1964), 231-238. MR 29 #5116.

[50] ITÔ, S. An elementary proof of a fundamental theorem on normed rings. *Sûgaku 14* (1962/63), 108-110. (Japanese). MR 26 #4200.

[51] KAMETANI, S. An elementary proof of the fundamental theorem of normed fields. *J. Math. Soc. Japan 4* (1952), 96-99. MR 14, 240.

[52] KAPLANSKY, I. Topological rings. *Bull. Amer. Math. Soc. 54* (1948), 809-826. MR 10, 179.

[53] —— Normed algebras. *Duke Math. J. 16* (1949), 399-418. MR 11, 115.

[54] LE PAGE, C. Sur quelques conditions entraînant la commutativité dans les algèbres de Banach. *C. R. Acad. Sci. Paris, Ser. A-B, 265* (1967), A235-A237. MR 37 #1999.

[55] LORCH, E. R. The theory of analytic functions in normed Abelian vector rings. *Trans. Amer. Math. Soc. 54* (1943), 414-425. MR 5, 100.

[56] MARKUSHEVICH, A. I. *Theory of functions of a complex variable.* Trans. by Richard A. Silverman, Vol. I, II. Prentice-Hall, Englewood Cliffs, 1965. MR 30 #2125, MR 31 #5965.

[57] MAZUR, S. Sur les anneaux linéaires. *C. R. Acad. Sci. Paris 207* (1938), 1025-1027.

[58] MOCANU, Gh. A remark on a commutativity criterion in Banach algebras. (Romanian. French summary) *Stud. Cerc. Mat. 21* (1969), 947-952. MR 43 #5311.

[59] —— Sur quelques critères de commutativité pour une algèbre de Banach. *An Univ. Bucureşti, Mat.-Mec. 20* (1971), 127-129. MR 46 #9739.

[60] —— On the spectral radius of an element of a Banach algebra. *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) 17* (65) (1973), 413-416 (1975). MR 52 #1310.

[61] MIYANAGA, Y. A note on Banach algebras. *Proc. Japan Acad. 32* (1956), 176. MR 17, 1112.

[62] MÜLLER, V. On the discontinuity of the spectral radius in Banach algebras. *Commentationes math. Univ. Carolinae 18* (1977), 591-598. Zbl. 358.46032.

[63] NAIMARK, M. A. *Normed Rings.* Second English edition, Groningen, The Netherlands, 1964. MR 34 #4928.

[64] ONO, T. Elementary proof of the basic theorem of normed rings. *Sûgaku 9* (1957/58), 236. (Japanese) MR 22 #5904.

[65] OSTROWSKI, A. Über einige Lösungen der Funktionalgleichung $\varphi(x)\varphi(y) = \varphi(xy)$. *Acta Math. 41* (1918), 271-284.

[66] PALAIS, R. S. The classification of real division algebras. *Amer. Math. Monthly 75* (1968), 366-368. MR 37 #4119.

[67] PONTRJAGIN, L. Über stetige algebraische Koerper. *Ann. of Math. 33* (1932) 163-174.

[68] —— *Topological Groups.* Princeton U. Press, Princeton, N. J., 1939. MR 1, 44.

[69] PTÁK, V. Derivations, commutators and the radical. *Manuscripta Math. 23* (1978), 355-362.

[70] PTÁK, V. and J. ZEMÁNEK. On uniform continuity of the spectral radius in Banach algebras. *Manuscripta Math. 20* (1977), 177-189. MR 56 #1065.

[71] RAMASWAMI, V. Normed algebras, isomorphism, and the associative postulate. *J. Indian Math. Soc. (N. S.) 14* (1950), 47-64. MR 12, 421.

[72] RICKART, C. E. An elementary proof of a fundamental theorem in the theory of Banach algebras. *Mich. Math. J. 5* (1958), 75-78. MR 20 #4786.

[73] —— *General Theory of Banach algebras.* Van Nostrand, Princeton, N. J., 1960. MR 22 #5903.

[74] SEGAL, I. E. Postulates for general quantum mechanics. *Ann. of Math. (2) 48* (1947), 930-948. MR 9, 241.

[75] SEID, H. A. A corollary to the Gelfand-Mazur theorem. *Amer. Math. Monthly 77* (1970), 282-283. MR 41 #4253.

[76] SHAFAREVICH, I. On the normalizability of topological fields. *Dokl. Akad. Nauk. SSSR 40* (1943), 133-135. MR 6, 164.

[77] SHILOV, G. An extension of maximal ideals. *C. R. (Dokalady) Acad. Sci. URSS (N. S.) 29* (1940), 83-84. MR 2, 314.

[78] SIMMONS, G. F. *Introduction to topology and modern analysis.* McGraw-Hill, New York, 1963. MR 26 #4145.

[79] SINCLAIR, A. M. *Automatic continuity of linear operators.* London Math. Soc. Lecture Note Series 21. Cambridge Univ. Press, Cambridge, 1976. Zbl. 313.47029.

[80] SLODKOWSKI, Z., W. WOJTYŃSKI, and J. ZEMÁNEK. A note on quasi-nilpotent elements of a Banach algebra. *Bull. Acad. Polon. Sci., Sér. Sci. math. astronom. phys., 25* (1977), 131-134. MR 55 #11043.

[81] SMILEY, M. F. Real Hilbert algebras with identity. *Proc. Amer. Math. Soc. 16* (1965), 440-441. MR 31 #591.

[82] SPICER, D. A commutativity theorem for Banach algebras. *Colloq. Math. 27* (1973), 107-108. MR 48 #9402.

[83] SRINIVASACHARYULU, K. Remarks on Banach algebras. *Bull. Soc. Roy. Sci. Liège 43* (1974), 523-525. MR 51 #11114.

[84] STONE, M. H. On the theorem of Gelfand-Mazur. *Ann. Polon. Math. 24* (1952), 238-240. MR 15, 132.

[85] STRZELECKI, E. Metric properties of normed algebras. *Bull. Acad. Polon. Sci. 10* (1962), 341-342. MR 26 #1767.

[86] —— Metric properties of normed algebras. *Studia Math. 23* (1963), 41-51. MR 34 #2620.

[87] —— Algebras under a minimal norm. *Colloq. Math. 11* (1963), 41-52. MR 28 #5351.

[88] —— Power-associative regular real normed algebras. *J. Austral. Math. Soc. 6* (1966), 193-209. MR 33 #6435.

[89] SUCIU, I. I. Eine natürliche Erweiterung der kommutativen Banach-algebra. *Rev. Math. Pures Appl. (Bucarest) 7* (1962), 483-491. MR 31 #1571.

[90] TAUSSKY, O. Sums of squares. *Amer. Math. Monthly 77* (1970), 805-830. MR 42 #3020.

[91] TAYLOR, A. E. The resolvent of a closed transformation. *Bull. Amer. Math. Soc. 44* (1938), 70-74.

[92] TILLER, W. *P*-commutative Banach *-algebras. *Trans. Amer. Math. Soc. 180* (1973), 327-336. MR 48 #877.

[93] TORNHEIM, L. Normed fields over the real and complex fields. *Mich. Math. J. 1* (1952), 61-68. MR 14, 131.

[94] TURPIN, P. Sur une classe d'algèbres topologiques. *C. R. Acad. Sci. Paris 263* (1966), A436-A439. MR 34 #6566.

[95] Urbanik, K. Remarks on ordered absolute-valued algebras. *Colloq. Math. 11* (1963), 31-39. MR 28 #5350.

[96] Urbanik, K. and F. Wright. Absolute-valued algebras. *Proc. Amer. Math. Soc. 11* (1960), 861-866. MR 22 #11021.

[97] Vesentini, E. On Banach algebras satisfying a spectral maximum principle. *Ann. Scuola Norm. Sup. Pisa, (3) 26* (1972), 933-943. MR 50 #14228.

[98] Williams, J. P. On commutativity and the numerical range in Banach algebras. *J. Functional Analysis 10* (1972), 326-329. MR 50 #14229.

[99] Wright, F. Absolute-valued algebras. *Proc. Nat. Acad. Sci. USA 39* (1953), 330-332. MR 14, 944.

[100] Yood, B. On the extension of modular maximal ideals. *Proc. Amer. Math. Soc. 14* (1963), 615-620. MR 27 #1848.

[101] Zelazko, W. On a certain class of topological division algebras. *Bull. Acad. Polon. Sci. 7* (1959), 201-203. MR 21 #3779.

[102] —— On the locally bounded and $m$-convex topological algebras. *Studia Math. 19* (1960), 333-356. MR 23 #A4033.

[103] —— A theorem on $B$-division algebras. *Bull. Acad. Polon. Sci. 8* (1960), 373-375. MR 23 #Å3198.

[104] —— *Metric generalizations of Banach algebras*. Dept. of Math., Yale University (Lecture notes), 1964.

[105] —— Metric generalizations of Banach algebras. *Rozprawy Math. 47* (1965), 70 pp. MR 33 #1752.

[106] —— On generalized topological divisors of zero in $m$-convex locally convex algebras. *Studia Math. 28* (1966), 9-16. MR 34 #3362.

[107] —— On generalized topological divisors of zero in real $m$-convex algebras. *Studia Math. 28* (1966/67), 241-244. MR 35 #7128.

[108] —— On topological divisors of zero in $p$-normed algebras without unit. *Colloq. Math. 16* (1967), 231-234. MR 35 #3442.

[109] —— *Banach Algebras*. North Holland, Amsterdam, 1973.

[110] Zemánek, J. Concerning spectral characterizations of the radical in Banach algebras. *Commentationes math. Univ. Carolinae 17* (1976), 689-691. MR 55 #1070.

[111] —— A note on the radical of a Banach algebra. *Manuscripta Math. 20* (1977), 191-196. MR 55 #8799.

[112] —— Properties of the spectral radius in Banach algebras. Banach Center Publications. Vol. *Spectral Theory*, to appear.

[113] Khasbardar, S. and Thakare, N. Commutativity in a Banach algebra. *Bollettino U.M.I. (5) 15-A* (1978), 581-584.

[114] Warner, S. A new approach to Gelfand-Mazur theory and the extension theorem. *Michigan Math. J. 26* (1979), 13-17.

*(Reçu le 9 mars 1979)*

V. A. Belfi
Robert S. Doran

Department of Mathematics
Texas Christian University
Fort Worth, Texas 76129
U.S.A.