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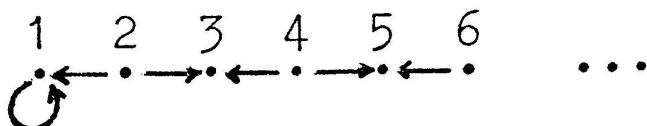
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$Z_{XXX} \vee XXX \notin M_{\{J, K\}}$  and so  $M_{\{J, K\}}$  is infinite, since for the infinite graph shown below:



$(JK)^m$  adds all edges  $\langle i, j \rangle$  with  $i, j \leq 2m$

and  $(JK)^m J$  adds all edges  $\langle i, j \rangle$  with  $i, j \leq 2m + 1$

Therefore  $J, JK, JKJ, \dots$  are all distinct.

#### 4. GENERALIZED ALGORITHM FOR POWER-GROUP ALGEBRAS

To elucidate the correctness of the algorithm and to encompass some more general applications we need to generalize from the  $\{0, 1\}$  Boolean algebra to a slightly richer structure. The *power-group algebra*  $P(G)$  is a structure defined from an arbitrary group  $G$ . The elements of  $P(G)$  are the subsets of  $G$ ; the operations we require are *union* ( $\cup$ ), *complex product*:

$$ab = \{gh \mid g \in a, h \in b\} \quad \text{for } a, b \subseteq G$$

and *converse*:

$$\bar{a} = \{g^{-1} \mid g \in a\}$$

$P(G)$  is a monoid with respect to product with identity  $\lambda = \{\text{identity}_G\}$ . As before we shall be considering matrices over the structure, with matrix product and union defined in the obvious way from product and union in  $P(G)$ , and matrix *converse* defined by

$$(\bar{A})_{ij} = \bar{A}_{ji}$$

The key properties of power-group algebras which are needed are given below

LEMMA. Let  $a, b$  be elements and  $A, B$  matrices

- (i)  $\bar{\bar{a}} = a$ ;  $\bar{\bar{A}} = A$
- (ii)  $\bar{ab} = \bar{b}\bar{a}$ ;  $\bar{AB} = \bar{B}\bar{A}$
- (iii) if  $a \neq \emptyset$  then  $a\bar{a} \supseteq \lambda$ ;  $A\bar{A}A \supseteq A$

*Proof.* We prove only (iii). The first part is immediate and has the consequence that  $\bar{a}\bar{a}a \supseteq a$  for all  $a$ . For the second part

$$(A\bar{A}A)_{ij} \supseteq A_{ij} \bar{A}_{ji} A_{ij} = A_{ij} \bar{A}_{ij} A_{ij} \supseteq A_{ij}.$$

□

We observe that the  $\{0, 1\}$  Boolean algebra is the power-group algebra corresponding to the trivial one-element group. Other groups we shall use are  $\langle \mathbf{Z}_k, + \rangle$  and  $\langle \mathbf{R}, + \rangle$ , the integers modulo  $k$  and the reals.

The operators  $*$ ,  $+$ ,  $V$  and  $W$  are defined just as before for matrices and elements. In the Boolean case we had the trivial results

$$\bar{a} = a^+ = a^W = a$$

and

$$a^* = a^V = 1$$

In the general case we must augment the algorithm a little. Suppose for example there are edges labelled  $a, b$  from  $i$  to  $j$  and  $k$  respectively, and a self-loop at  $i$  labelled  $c$ . Then the label of the edge from  $j$  to  $k$  must eventually receive a term corresponding to the indirect paths from  $j$  to  $k$  i.e.

$$\bar{a}(\bar{a}\bar{a} \vee c \vee \bar{c} \vee \bar{b}\bar{b})^* b$$

The generalized form of  $\psi_i$  is:

$$A_{i*} := A_{ii}^V A_{i*}$$

$k := i + 1$  step 1 until ( $k = n$  or  $A_{ik} \neq \emptyset$ )

if  $A_{ik} \neq \emptyset$  then  $A_{k*} := A_{k*} \cup \bar{A}_{ik} A_{i*}$

The programs for  $\psi'_i, \Psi$  and  $\Psi'$  are analogous. They simplify to the programs of section 2 in the Boolean case.

The question of generalization could have been tackled axiomatically. Suppose  $i < j < k$  and  $A_{ij}, A_{ik} \neq \emptyset$ . If row  $i$  were sent directly to row  $k$  we would have  $\bar{A}_{ik} A_{i*}$ , whereas via  $j$  we get

$$\bar{A}_{ij} \bar{A}_{ik} \cdot \bar{A}_{ij} A_{i*} = \bar{A}_{ik} A_{ij} \bar{A}_{ij} A_{i*}$$

and we seem to require  $A_{ij} \bar{A}_{ij} \geq \lambda$  for correctness. A power-group algebra seems the only structure of possible interest with this property.

Suppose that  $A$  is a matrix over  $P(G)$  and we compute  $A^{RQ} = A^V$ .  $A^V_{ij}$  is the set of elements of  $G$  which are the product of labels from a (weak) path from  $i$  to  $j$  in the graph corresponding to  $A$ . Each diagonal element

$A^V_{ii}$  is a subgroup of  $G$  and two diagonal elements in the same (weak) component are conjugate since we may show:

$$A^V_{ii} \cdot g = g \cdot A^V_{jj} = A^V_{ij} \quad \text{for any } g \in A^V_{ij}$$

In particular if all circuits on non-empty edges correspond to the group identity then in  $A^V$  each entry has at most one element. For example given a matrix  $A$  over  $\langle \mathbf{R}, + \rangle$  we determine from  $A^V$  for each component of the graph whether there are (weak) circuits with non-zero sums. If all circuits sum to zero then the graph is a “potential” graph, i.e. there is a function  $\text{pot} : \mathbf{R} \rightarrow \{\text{vertices}\}$  such that each edge  $\langle u, v \rangle$  has the value  $(\text{pot}(v) - \text{pot}(u))$ . Similarly over  $\langle \mathbf{Z}_k, + \rangle$  we may determine whether each weak circuit of a directed graph has zero sum where forward and backward edges are accounted +1 and -1 respectively. Naturally we may find it convenient in some cases to hold only a homomorphic image of  $P(G)$  for the computations e.g.

$$\begin{aligned} h(\emptyset) &= \emptyset \\ h(\{g\}) &= g \\ h(a) &= \omega \quad \text{when } |a| > 1. \end{aligned}$$

## 5. PROOFS OF CORRECTNESS

An operator  $\phi$  is *monotonic* if  $A \subseteq B$  implies  $A^\phi \subseteq B^\phi$ .  $\psi_i$  is *not* a monotonic operator but rather surprisingly  $\Psi$  is. To simplify our proofs we introduce several monotonic operators. Define  $\phi_i$  by the program

$$\begin{aligned} A_{i*} &:= A_{ii}^V A_{i*} \\ \text{for } k &:= i + 1 \text{ step } 1 \text{ until } n \text{ do } A_{k*} := A_{k*} \cup \overline{A_{ik}} A_{i*} \end{aligned}$$

and  $\phi'_i$  analogously using “ $i - 1$  step  $-1$  until 1”. Both are obviously monotonic.  $\Phi$  and  $\Phi'$  are defined from  $\phi_i$  and  $\phi'_i$  in a similar way to  $\Psi$  and  $\Psi'$ . Although  $\Psi \subseteq \Phi$  is evident, the following result is not.

**THEOREM 5.**  $\Psi = \Phi$  (and  $\Psi' = \Phi'$ ).

*Proof.* Consider  $\psi_i$  applied to an arbitrary matrix  $A$  and suppose it selects the index  $k$  with  $A_{ik} \neq \emptyset$ . (If no index is selected then of course  $A^{\psi_i} = A^{\phi_i}$ ). We verify that: