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that of product [2]) is no greater than that of T , to within a constant factor. However we have been unable to show the converse.

Open Problem 1. Is there an $O(n^2)$ matrix-based algorithm for the T' -closure?

3. THE QUADRATIC MONOID

To satisfy our curiosity we investigated the monoid generated by the composition of closures corresponding to polynomials of degree at most two. For any set of transformations E let M_E be the monoid generated by compositions of elements of E . For any polynomial $P(X, \bar{X})$, define $Z_P : A \rightarrow (\mu X. X \geqslant A \vee P(X, \bar{X}))$ and then

$$\Pi_r = \{ Z_P \mid \deg(P) \leqslant r \}.$$

THEOREM 3. $M_{\Pi_2} = M_{\{R, S, Q, Q', T, T'\}}$ and the monoid is finite.

Proof. The equality follows from the finiteness since

$$\begin{aligned} Z_{P_1 \vee P_2} &= \bigvee_m (Z_{P_1} \cdot Z_{P_2})^m \\ &\in M_{\{Z_{P_1}, Z_{P_2}\}} \quad \text{if this is finite.} \end{aligned}$$

$M_{\{R, S, Q, Q', T, T'\}}$ is examined explicitly below and is found to contain exactly fifty elements. \square

We write A for the monoid identity given by $A^A = A$ and $[Z_1, \dots, Z_k]$ for the closure $\bigvee_m (Z_1 \vee \dots \vee Z_k)^m$. Together with the obvious idempotencies of closures we have the following sufficient defining relations.

$$\begin{aligned} W &\stackrel{\text{def}}{=} [S, Q, Q', T, T'] \\ &= QQ' = Q'Q = QT' = Q'T' = SQ = SQ' = ST = ST' \\ V &\stackrel{\text{def}}{=} [R, S, Q, Q', T, T'] = WR = RQ = RQ' = RT' \\ QT &= [Q, T] \quad Q'T = [Q', T] \\ T'Q &= T'TQ = T'QT \quad T'Q' = T'TQ' = T'Q'T \\ TT' &= T'T * \stackrel{\text{def}}{=} RT = TR \quad RS = SR \end{aligned}$$

The closures in the monoid are

$$\begin{aligned} V &\quad : \quad A^V = (\bar{A} \vee A)^* \\ W &\quad : \quad A^W = (\bar{A} \vee A)^+ \end{aligned}$$

$$\begin{aligned}
 [Q, T] : A^{[Q, T]} &= A^V \cdot A \\
 [Q', T] : A^{[Q', T]} &= A \cdot A^V \\
 [T, T'] : A^{[T, T']} &= A \vee A^V \cdot (AA \vee \bar{A}\bar{A}) \cdot A^V \\
 *, [R, S], R, S, Q, Q', T, T' \text{ and } A.
 \end{aligned}$$

The monoid can be counted after expressing its elements in a canonical form by the following rules.

- (i) Using $RS = SR, RT = TR, RQ = RQ' = RT' = QQ'R$, we can bring any occurrence of R to the end of the product
- (ii) Using $SQ = SQ' = ST = ST' = QQ'$, we can assume that any S occurs at the end of the rest of the product
- (iii) $QT' = Q'T' = T'QQ'$ and $TT' = T'T$ allow us to bring any T' to the front of the remainder.
- (iv) The elements generated by Q, Q', T are found to be

$$A, Q, Q', T, TQ, TQ', QT, Q'T, W$$

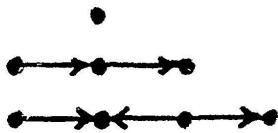
Prefixing these with T' yields only 4 new elements

$$T', T'Q, T'Q', T'T$$

- (v) The 50 elements of the monoid are given by

$$\begin{aligned}
 \{ A, Q, Q', T, T', TQ, TQ', QT, Q'T, T'Q, T'Q', T'T \} \cdot \{ A, R, S, SR \} \\
 \cup \{ W, V \}
 \end{aligned}$$

These elements are distinguishable by their effect on the graph



The “fast” monoid generated by the $O(n^2)$ operations R, S, Q, Q' has only 14 elements

$$\{ A, Q, Q' \} \cdot \{ A, R, S, SR \} \cup \{ W, V \}$$

Of computational interest are the relations

$$RQ = V \quad \text{and} \quad QQ' = SQ = W$$

which yield efficient ways to compute these common closures. Note that in some contexts the Q' closure may be more rapid to compute than S .

We illustrate some of the proofs for the results above.

THEOREM 4.

CANCELLATION LEMMA. *For all A , $A\bar{A}A \geq A$.*

Proof. $(A\bar{A}A)_{ij} \geq A_{ij}\bar{A}_{ji}A_{ij} = A_{ij}A_{ij}A_{ij} \geq A_{ij}$ □

- (i) $RQ = V$
- (ii) $QQ' = W$
- (iii) $[Q, T] = QT$ and $A^{QT} = A^V \cdot A$

Proof.

- (i) If we show that $RQ \geq S$ the result follows easily. But

$$A^{RQ} \geq (I \vee A)^Q \geq A \vee \bar{A}I = A \vee \bar{A} = A^S$$

- (ii) Again the only non-trivial part is that $QQ' \geq S$

$$A^{QQ'} \geq (A \vee \bar{A}A)^{Q'} \geq A \vee \bar{A}A \cdot \bar{A} \geq A^S$$

by the Cancellation Lemma.

- (iii) By inspection, $A^{[Q, T]} \leq A^V \cdot A$

However,

$$A^V \cdot A = A^* \cdot \bar{A}A^V A \vee A^* \cdot A \leq (A^Q)^T \leq A^{[Q, T]}$$
□

One of the harder results to prove is that $TT' = T'T$. We leave it as an exercise for the reader.

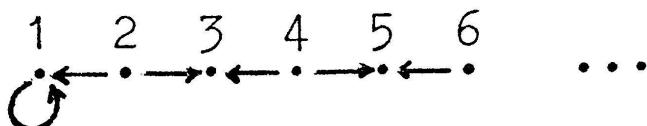
We have found that each mapping in Π_2 is defined by a regular set over $\{A, \bar{A}\}$, however in Π_3 there are «non-regular» mappings, e.g. $Z_{xxx} \vee x_{xx}$.

The finiteness of M_{Π_2} does not persist for large r . We show by the example below that M_{Π_4} is infinite.

Open Problem 2. Is M_{Π_3} finite?

Example. Let $J = Z_{xxxx}, K = Z_{xxxxx}$

$Z_{XXX} \vee XXX \notin M_{\{J, K\}}$ and so $M_{\{J, K\}}$ is infinite, since for the infinite graph shown below:



$(JK)^m$ adds all edges $\langle i, j \rangle$ with $i, j \leq 2m$

and $(JK)^m J$ adds all edges $\langle i, j \rangle$ with $i, j \leq 2m + 1$

Therefore J, JK, JKJ, \dots are all distinct.

4. GENERALIZED ALGORITHM FOR POWER-GROUP ALGEBRAS

To elucidate the correctness of the algorithm and to encompass some more general applications we need to generalize from the $\{0, 1\}$ Boolean algebra to a slightly richer structure. The *power-group algebra* $P(G)$ is a structure defined from an arbitrary group G . The elements of $P(G)$ are the subsets of G ; the operations we require are *union* (\cup), complex *product*:

$$ab = \{gh \mid g \in a, h \in b\} \quad \text{for } a, b \subseteq G$$

and *converse*:

$$\bar{a} = \{g^{-1} \mid g \in a\}$$

$P(G)$ is a monoid with respect to product with identity $\lambda = \{\text{id}_G\}$. As before we shall be considering matrices over the structure, with matrix product and union defined in the obvious way from product and union in $P(G)$, and matrix *converse* defined by

$$(\bar{A})_{ij} = \bar{A}_{ji}$$

The key properties of power-group algebras which are needed are given below

LEMMA. Let a, b be elements and A, B matrices

- (i) $\bar{\bar{a}} = a$; $\bar{\bar{A}} = A$
- (ii) $\bar{ab} = \bar{b}\bar{a}$; $\bar{AB} = \bar{B}\bar{A}$
- (iii) if $a \neq \emptyset$ then $a\bar{a} \supseteq \lambda$; $A\bar{A}A \supseteq A$