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4. Applications

Let us start by deriving some results which could also be obtained from the theorems in [3, 4, 6] mentioned in the introduction. Abreviating $x = x_1, ..., x_n, y = y_1, ..., y_k$, consider $\Omega = \overline{F(x, y)}, K = \overline{F(x)}, E = \overline{F(y)}$. Then E and K are linearly disjoint over \overline{F} (see e.g. [1], p. 203). Taking $k = 1, e_i = y_1^i, 1 \le i \le n$, we see that any computation of $f(y_1) = x_1 y_1 + ... + x_n y_1^n$ in $(\Omega, E \cup K)$ requires $\lceil \frac{n}{2} \rceil M/D$ that count even if we disregard a M/D by an element $g \in \overline{F}$. Thus any preprocessing using algebraic functions $\alpha_1, ...$ in x and algebraic functions $\beta_1, ...$ in y, cannot save more than $\frac{n}{2} M/D$.

Taking k = n, we get a similar result for $x_1 y_1 + ... + x_n y_n$.

In [6] Winograd has considered the computation of the product Ax where $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ is an $m \times n$ matrix and x is the column vector $x = (x_1, ..., x_n)$. Computing Ax means, of course, computing the forms $a_{i1} x_1 + ... + a_{in} x_n, 1 \le i \le m$. In our notations assume that $a_{ij} \in E$, $x_1, ..., x_n \in K$. Denote the column vectors of A by $v_1, ..., v_n$, thus $v_j \in E^m$.

We say that $\dim_{E^m/F^m} (v_1, ..., v_n) = r$, if r is the largest integer such that for some subset $\{i_1, ..., i_r\} \subseteq \{1, ..., n\}$

 $g_1 v_{i_1} + \ldots + g_r v_{i_r} \in F^m, g_i \in F$ implies $g_i = 0, 1 \leq i \leq r$.

Winograd [6] assumes that $\dim_{E^m/F^m} (v_1, ..., v_n) = r$, and that $F \subseteq \mathbb{C}$ the field of complex numbers. Furthermore K is a field such that $F(x_1, ..., x_n) \subseteq K$ and K is embeddable in a field of continuous (except for isolated points) functions $f(x_1, ..., x_n)$ into \mathbb{C} which vanish only at isolated points; similarly $F(y_1, ..., y_m) \subseteq E$, and E is embeddable in a field of functions $g(v_1, ..., y_m)$ with the above properties. Under these conditions, an algorithm for Ax requires at least $\lceil \frac{r}{2} \rceil M/D$ that count.

In purely algebraic terms we can state and prove the following theorem.

THEOREM 2. Let $A = (a_{ij})$ be an $m \times n$ matrix with $a_{ij} \in E$ and let $x_1, ..., x_n \in K$ be algebraically independent over F. Denote the columns of A by $v_1, ..., v_n$. If E and K are linearly disjoint over F, and if

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 $\dim_{E^m/F^m}(v_1, ..., v_n) = r$, then any algorithm π in $(\Omega, E \cup K)$ which computes Ax has at least $\lceil \frac{r}{2} \rceil M/D$ that count.

Proof. Using vector notation, computing Ax means computing all coordinates of the sum

(8)
$$x_1v_1 + \ldots + x_nv_n = w$$
.

We may assume that r = n. Otherwise let without loss of generality $v_1, ..., v_r, r < n$, be vectors which are independent mod F^m over F. Then for $r < j \le n$

$$v_{j} = g_{j1}v_{1} + \ldots + g_{jr}v_{r} + u_{j}, \ g_{ji} \in F, \ u_{j} \in F^{m}.$$

Hence, from (8),

$$w = (x_1 + g_{r+1,1}x_{r+1} + \dots + g_{n1}x_n)v_1 + \dots + x_{r+1}u_{r+1} + \dots + x_nu_n$$

= $z_1v_1 + \dots + z_rv_r + u$,

where $u \in K^m$. Now the computation in $(\Omega, E \cup K)$ of u costs nothing, and the $z_1, ..., z_r \in K$ are algebraically independent over F. So we have the conditions of the theorem with r = n.

Assume from now on that $v_1, ..., v_n$ are independent mod F^m over F. Let $e_0 = 1, e_1, ..., e_p$ be elements in E which are linearly independent over F, such that every a_{ij} (the *i*-th component of v_j), $1 \le i \le m, 1 \le j \le n$, is a linear combination of $e_0, ..., e_p$ with coefficients in F. Each v_j can be split $v_j = u_j + w_j$, where $u_j \in F^m$, and every coordinate of w_j is a linear combination of just $e_1, ..., e_p$ with coefficients in F. Thus $w = x_1 w_1 + ... + x_n w_n + u$, where $u \in K^m$, and computing $x_1 w_1 + ... + x_n w_n$ in $(\Omega, E \cup K)$ takes as many M/D that count as does computing w.

Because $v_1, ..., v_n$ are linearly independent mod F^m over F, we have that $w_1, ..., w_n$ are linearly independent over F. Consider the sum $Z_1 w_1 + ... + Z_n w_n$, where $Z_1, ..., Z_n$ are variables ranging over Ω . Writing the *i*-th coordinate of w_k as a linear combination $\sum_{j=1}^{p} g_{ijk} e_j$ and rearranging, we get

(9)
$$Z_1 w_1 + \ldots + Z_n w_n = [L_{i1}(Z) e_1 + \ldots + L_{ip}(Z) e_p]_{1 \le i \le m}$$

where $L_{ij}(Z) = \sum_{k=1}^{n} g_{ijk} Z_{k}$.

We claim that among the $L_{ij}(Z)$, $1 \le i \le m$, $1 \le j \le p$, there are *n* forms which are linearly independent. By this we mean that the rows of

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coefficients of these *n* forms are linearly independent over *F*. Otherwise there are $h_1, ..., h_n \in F$, not all 0, so that the substitution $Z_1 = h_1, ..., Z_n$ $= h_n$ yields $L_{ij}(h) = 0, 1 \le i \le m, 1 \le j \le p$. By (9) we now have $h_1 w_1 + ... + h_n w_n = 0$, contradicting the linear independence of $w_1, ..., w_n$ over *F*.

Let $L_{i_1j_1}(Z), ..., L_{i_nj_n}(Z)$ be such a system of *n* independent forms. Then $d_{i_1j_1} = L_{i_1j_1}(x_1, ..., x_n), ..., d_{i_nj_n} = L_{i_nj_n}(x_1, ..., x_n)$ are algebraically independent over *F*. This is because $x_1, ..., x_n$ is the unique solution of the regular system of linear equations

$$L_{i_e j_e}(Z_1, ..., Z_n) = d_{i_e j_e}, \quad 1 \leq e \leq n$$

Thus, finally

(10)
$$x_1 w_1 + \dots + x_n w_n = [d_{i1}e_1 + \dots + d_{ip}e_p]_{1 \le i \le m}$$

with $d_{ij} \in K$, and the degree of transcendence of the d_{ij} over F is n. So, by Theorem 1, at least $\lceil \frac{n}{2} \rceil M/D$ that count are needed to compute (10), and hence to compute (8) in $(\Omega, E \cup K)$.

For the next application let $x_1, ..., x_n$ be algebraically independent over F and put $\Omega = \overline{F(x_1, ..., x_n)}, E = \overline{F}, K = F(x_1, ..., x_n)$. Then, by an argument like the one used in the first example after the statement of Theorem 1, E and K are linearly disjoint over F. Therefore Theorem 1 implies that for any $\omega \in E$ of degree at least n + 1 over F the computation of

(11)
$$\omega x_1 + \ldots + \omega^n x_n$$

in $(\Omega, E \cup K)$ requires at least $\lceil \frac{n}{2} \rceil M/D$. Note that now we have a result about substitution of a specific algebraic number in a polynomial. We allow any rational preprocessing of the coefficients and any algebraic preprocessing of the argument ω .

Next we show that no finite number of algebraic functions of $x_1, ..., x_n$ simplifies the computation of (11) for all algebraic ω of degree n + 1 over the rationals **Q**. Since any particular preprocessing of $x_1, ..., x_n$ by algebraic functions involve just a finite number of such functions, we essentially conclude that algebraic preprocessing of $x_1, ..., x_n$ in (11), as well as the ω (ω now depends on the chosen preprocessing of the x_i of course), does not reduce the number of M/D that count below $\lceil \frac{n}{2} \rceil$. Specifically

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THEOREM 3. Let

$$G = \mathbf{Q}(x_1, ..., x_n), \Omega = \overline{G}, a_1, ..., a_q \in \Omega, K = G(a_1, ..., a_q)$$

and $F = \mathbf{Q}$. There exists an element $\omega \in \overline{\mathbf{Q}}$ of degree n + 1 over \mathbf{Q} such that any computation π for (11) in $(\Omega, \overline{\mathbf{Q}} \cup K)$ must have at least $\lceil \frac{n}{2} \rceil M/D$ that count.

Proof. Define $F_1 = \overline{\mathbf{Q}} \cap K$. We shall prove slightly more than stated, namely that for a suitable $\omega \in \overline{\mathbf{Q}}$, computation of (11) in $(\Omega, \overline{\mathbf{Q}} \cup K)$ requires at least $\lceil \frac{n}{2} \rceil M/D$ that count even if we disregard M/D by a $g \in F_1$. The diagram of fields is

$\mathbf{Q}(\mathbf{x}_1, .$	$, x_n$)
UĮ	\mathcal{U}
$\overline{\mathbf{Q}}$	K
\mathcal{U}_{l}	Uj
$F_1 = \overline{0}$	$\overline{\mathbf{Q}} \cap K$
UI	
F =	Q

Notice that $\overline{\mathbf{Q}} = \overline{F}_1$ and $\overline{F}_1 \cap K = F_1$. This implies that $\overline{\mathbf{Q}}$ and K are linearly disjoint over F_1 . Namely let $e_1, \ldots, e_q \in \overline{F}_1$ be independent over F_1 . Choose a primitive element $e \in \overline{F}_1$, of degree m over F say, such that $e_1, \ldots, e_q \in F_1$ (e), and let $f(X) \in F_1[X]$ be the minimal polynomial of eover F_1 . Assume $f = f_1 f_2$ in K[X]. Since the coefficients of f_1, f_2 are algebraic over F_1 and since $\overline{F}_1 \cap K = F_1$ we obtain $f_1, f_2 \in F_1[X]$. Therefore f is irreducible in K[X] and hence the elements $1, e, \ldots, e^{m-1}$ are linearly independent over K. By linear algebra it follows that e_1, \ldots, e_q are

The degree $[F_1:\mathbf{Q}]$ is at most $[K:\mathbf{Q} (x_1, ..., x_n)]$ hence finite. This implies that for any *n* there exists an algebraic number $\omega \in \overline{\mathbf{Q}}$ of degree n + 1 over \mathbf{Q} which retains the degree n + 1 over F_1 . For this ω the statement in the theorem holds true as a consequence of Theorem 1.