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### 3. PROOF OF THEOREM

Put  $e_0 = 1$  and let  $(e_i)_{i < \kappa}$  ( $\kappa$  some cardinal) be an extension of  $e_0, e_1, \dots, e_p$  to an  $F$ -base of  $E$ . By linear disjointness,  $(e_i)_{i < \kappa}$  is also a  $K$ -base of the  $K$ -algebra  $K[E]$ . Since for  $i, j < \kappa$   $e_i e_j = \sum_k g_{ijk} e_k$  for suitable  $g_{ijk} \in F$  we have:

(5) If  $(\sum a_i e_i)(\sum b_j e_j) = \sum c_k e_k$  where  $a_i, b_j, c_k \in K$

(and the sums are finite of course) then

$$c_k = \sum_{i,j} a_i b_j g_{ijk} \in F[\{a_i\} \cup \{b_j\}],$$

Any element  $r \in KE$ , the quotient field of  $K[E]$ , can be written in the form

$$r = \frac{\sum a_i e_i}{\sum b_j e_j} + c$$

where  $a_i, b_j, c \in K$ , not all  $b_j = 0$ . Such a representation of  $r$  will be called a canonical representation, and the  $a_i$ 's and  $b_j$ 's are the coefficients of the given representation. Note that the canonical representation is not unique.

**LEMMA.** *If  $r_1, \dots, r_n$  is the sequence of results of some computation in  $(\Omega, E \cup K)$  using  $s M/D$  that count then there are  $2s$  elements  $\alpha_1, \dots, \alpha_{2s} \in K$  such that each  $r_v \neq u$ ,  $1 \leq v \leq n$ , has a canonical representation all of whose coefficients are in  $F[\alpha_1, \dots, \alpha_{2s}]$ .*

The proof is by induction on  $n$ . The case  $n = 0$  being trivial assume  $n > 0$ .

If  $r_n \in E \cup K$  then obviously  $r_n$  has a canonical representation with coefficients in  $F$ , so the claim follows from the induction hypothesis. The same applies if  $r_n = u$ .

Next assume that  $r_n \in \Omega$  is the result of a non-counting operation, i.e.  $r_n = r_\mu \pm r_v$  for some  $\mu, v < n$  or  $r_n$  is the result of a  $M/D$  where one of the factors or the denominator is a  $g \in F$ . Let us consider the case  $r_n = r_\mu + r_v$ , the other cases are similar. Choose  $\alpha_1, \dots, \alpha_{2s} \in K$  and canonical representations

$$r_\mu = \frac{A}{B} + c, \quad r_v = \frac{A'}{B'} + c'$$

according to the induction hypothesis. Then, by (5), the coefficients of the canonical representation

$$r_n = \frac{AB' + A'B}{BB'} + (c + c')$$

also lie in  $F[\alpha_1, \dots, \alpha_{2s}]$ .

Finally let  $r_n = r_\mu \cdot r_\nu$  ( $r_n = r_\mu/r_\nu$  resp.),  $r_n \in \Omega$ . Then, again by (5), the coefficients of the representation

$$r_n = \frac{(A + cB)(A' + c'B')}{BB'} \quad \left( r_n = \frac{(A + cB)B'}{(A' + cB')B} \text{ resp.} \right)$$

lie in  $F[\alpha_1, \dots, \alpha_{2s-2}, c, c']$  where  $\alpha_1, \dots, \alpha_{2s-2} \in K$  are provided by induction hypothesis. Putting  $\alpha_{2s-1} = c, \alpha_{2s} = c'$  completes the induction.

*Proof of Theorem 1.* Assume that  $\pi$  computes the elements  $\sum_{j=1}^p d_{ij} e_j$ ,  $1 \leq i \leq m$ , in  $(\Omega, E \cup K)$  with  $s$  counting  $M/D$ . By the Lemma there exist  $\alpha_1, \dots, \alpha_{2s} \in K$  and canonical representations

$$(6) \quad \sum_{j=1}^p d_{ij} e_j = \frac{\sum_k a_{ik} e_k}{\sum_q b_{iq} e_q} + c_i, \quad 1 \leq i \leq m,$$

with coefficients  $a_{ik}, b_{iq} \in F[\alpha_1, \dots, \alpha_{2s}]$ . Now fix  $i$ . Multiplying (6) by the denominator gives

$$(7) \quad \left( \sum_q b_{iq} e_q \right) \left( -c_i e_0 + \sum_j d_{ij} e_j \right) = \sum_k a_{ik} e_k.$$

Multiplying out the left hand side and comparing the coefficients of each  $e_k$  on both sides (recall that  $e_0, e_1, \dots$  are independent over  $K$ ) we obtain, by using (5), a system  $\mathcal{S}$  of linear equations for the  $d_{ij}$ 's and  $c_i$  whose coefficients are  $F$ -linear forms of the  $b_{iq}$ 's. Now the equation (7) clearly determines the element  $-c_i e_0 + \sum_j d_{ij} e_j$  uniquely. Since the  $e_j$  are  $K$ -linear independent it follows that  $\mathcal{S}$  has a unique solution, and hence  $d_{ij}, c_i \in F(\alpha_1, \dots, \alpha_{2s})$ , by linear algebra. Since  $D$  has degree of transcendence  $t$  over  $F$  we obtain  $2s \geq t$ , i.e.  $s \geq \lceil \frac{t}{2} \rceil$ .

*Remark.* The method for handling divisions was proposed by Volker Strassen and we kindly thank him for this.