Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 26 (1980)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LINEAR DISJOINTNESS AND ALGEBRAIC COMPLEXITY

Autor: Baur, Walter / Rabin, Michael O.

Kapitel: 2. Basic concepts and the Main Theorem

DOI: https://doi.org/10.5169/seals-51078

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

proof are expressed in purely algebraic terms. In section 4 we apply Theorem 1 to obtain the known results on lower bounds, as well as new results which do not fall within the scope of previous methods.

2. Basic concepts and the Main Theorem

Let Ω be a field and S a subset of its elements. Following [5, 6], a (straight-line) algorithm or computation in (Ω, S) is a sequence π : $\pi(1), ..., \pi(l)$ where for each $1 \le k \le l$ we have $\pi(k) \in S$, or for some $i, j < k, \pi(k) = (+, i, j)$ or (-, i, j) or (-, i, j) or (/, i, j).

With π we associate the sequence r(1), ..., r(l) of the results of the computation π . The r(k) are all elements of $\Omega \cup \{u\}$. Define $r(1) = \pi(1) \in S$. Inductively, if r(1), ..., r(k-1) are already defined we set $r(k) = \pi(k)$ if $\pi(k) \in S$, r(k) = r(i) + r(j) if $\pi(k) = (+, i, j)$, etc. By convention, $r/0 = u + r = u \cdot r = ... = u$ for $r \in \Omega \cup \{u\}$.

We say that π computes the elements $a_1, ..., a_m \in \Omega$ if there exist $1 \le i_j \le l$, $1 \le j \le m$, so that for the results of π we have $r(i_j) = a_j$, $1 \le j \le m$.

In the sequel we shall be interested in fields $F \subseteq \Omega$ and two intermediate fields E, K. Thus

The following concept comes from the theory of fields and from algebraic geometry, see [1, 2].

Definition. The fields E and K are linearly disjoint over F if any $e_1, ..., e_m \in E$ which are linearly independent over F are also linearly independent over K, i.e. $\sum a_i e_i = 0$, $a_i \in K$, only if $a_i = 0$, $1 \le i \le m$.

As the definition stands, the fields E and K play different roles. It is however easy to see that the above definition implies the analogous statement with the roles of E and K interchanged. (See e.g. [1].)

Our theorem will be about computations π in $(\Omega, E \cup K)$. The fact that we permit using any $\alpha \in E \cup K$ at no computational cost captures, in an algebraic form, the idea of preprocessing.

We shall strengthen the contents of our lower bound results by disregarding those M/D used in a computation π where one of the factors or the denominator is a $g \in F$. An M/D-operation $\pi(k) = (\sigma, i, j)$ counts if $r(k) \neq u$ and either $\sigma = \cdot$ and $r(i), r(j) \notin F$, or $\sigma = /$ and $r(j) \notin F$.

Given $e_1, ..., e_p \in E$, we say that they are independent mod F over F if $\Sigma g_i e_i \in F$ and $g_i \in F$, $1 \le i \le p$, implies $g_i = 0$, $1 \le i \le p$.

With these concepts we can state our main result.

Theorem 1. Assume that E and K in (3) are linearly disjoint over F. Let $d_{ij} \in K$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p$, be such that the degree of transcendence of $D = \{d_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p\}$ over F is t. Let $e_1, ..., e_p \in E$ be linearly independent $\mod F$ over F. If π is any algorithm in $(\Omega, E \cup K)$ which computes all the m elements

$$d_{11}e_{1} + \dots + d_{1p}e_{p}$$

$$\vdots$$

$$d_{m1}e_{1} + \dots + d_{mp}e_{p}$$

then π has at least $\lceil \frac{t}{2} \rceil_{M/D}$ that count.

The proof will be given in section 3. Let us consider some preliminary examples.

In (3), let $\Omega = F(x_1, ..., x_n, y_1, ..., y_n)$ where $x_1, ..., y_n$ are algebraically independent over F, and let $E = F(y_1, ..., y_n)$, $K = F(x_1, ..., x_n)$. Then E and K are linearly disjoint over F. This can be seen as follows: Assume $\sum r_i(x) s_i(y) = 0$ is a nontrivial dependence relation, $r_i(x) \in K$, $s_i(y) \in E$. Multiplying by some $r(x) \in F[x_1, ..., x_n]$ we may assume that all $r_i(x) \in F[x_1, ..., x_n]$. Let m be a monomial in $x_1, ..., x_n$ occurring in at least one $r_i(x)$ and let $g_i \in F$ be the coefficient of m in $r_i(x)$. Then $\sum g_i s_i(y)$ is a nontrival dependence relation with coefficients from F.

So the conditions of Theorem 1 hold for the inner product $(x, y) = x_1 y_1 + ... + x_n y_n$ with t = n (and m = 1). Hence no algorithm π computing (x, y), even when allowed to use at no cost any rational functions $r(x_1, ..., x_n) \in K$, $s(y_1, ..., y_n) \in E$ can have fewer than $\lceil \frac{n}{2} \rceil M/D$ that count. Much stronger results on (x, y) will be given later, but we mention this

fact now as an illustration of the concepts and because Winograd's preprocessing is of the kind covered by this remark.

The need for the condition that the e_i are linearly independent mod F is clear. Otherwise if, say, m=1 and $e_i=g_i\,e_1+h_i,g_i,\,h_i\in F,\,2\leqslant i\leqslant p$ then

$$d_1 e_1 + \ldots + d_p e_p = (d_1 + g_2 d_2 + \ldots + g_p d_p) e_1 + h_2 d_2 + \ldots + h_p d_p.$$

Thus there is only one multiplication that counts.

It is not sufficient to require in Theorem 1 that $E \cap K = F$, even though this might seem to prevent a computation in $(\Omega, E \cup K)$ from "mixing" without cost elements from E with elements from K: Let Ω be the quotient field of the integral domain $F[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_1 + x_2y_2 + x_3y_3)$, and put $E = F(x_1, x_2, x_3) \subseteq \Omega$, $K = F(y_1, y_2, y_3) \subseteq \Omega$. In Ω , the elements x_1, x_2, x_3 are still algebraically independent over F, and similarly for y_1, y_2, y_3 . Also $E \cap K = F$. So the conditions of Theorem 1, with $E \cap K = F$ instead of linear disjointness, hold for $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$. But the computation of this sum requires no operation instead of 2M/D.

One might think that the condition of linear disjointness on E and K in Theorem 1 is already so strong that we could replace the degree of transcendence t by just the linear dimension. Thus if $e_1, ..., e_p \in K$ are linearly independent mod F over F and similarly for $d_1, ..., d_p \in K$, and E and K are linearly disjoint over F, does $\sum d_i e_i$ require at least $\lceil \frac{p}{2} \rceil M/D$ that count. The next example refutes this conjecture.

Denoting the algebraic closure of a field H by \overline{H} , let $\Omega = \overline{G(x,y)}$ where x,y are algebraically independent over G. Let n>1 and put $F=G(x^n,y^n), E=F(x), K=F(y)$. Clearly the F-base $1,x,...,x^{n-1}$ of E remains linearly independent over K. Hence, by linear algebra, E and K are linearly disjoint over F. Consider the element

$$\frac{1-x^ny^n}{1-xy}-1=xy+x^2y^2+\ldots+x^{n-1}y^{n-1}.$$

Obviously this element can be computed in $(\Omega, E \cup K)$ with 2 M/D.