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Autor: Libgober, A.
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§ 2. WEIL-MILGRAM QUADRATIC RECIPROCITY LAW

Let A_F denote the adèle group of the field F i.e. the group of infinite vectors $(\dots x_v \dots)$ where v runs through all valuations of F , x_v is an element of completion F_v , and all x_v except for finite number of v 's are integers. F is diagonally embedded in A_F . Let χ denote a character of A_F trivial on F . Let χ_0 be the following special character of such type. On the archimedean component v of the adèle $\{x_v\} \in A_F$, χ_0 takes the value $\exp(-2\pi i \text{Tr} x_v)$, and on a non-archimedean component $\exp(2\pi i \text{Tr} x_v)$. Here Tr denotes the absolute trace from the v -component F_v of A_F to $\mathbf{Q}_{\bar{v}}$ where \bar{v} is a valuation of \mathbf{Q} over which lies v . Recall that exponent of p -adic number a is defined as exponent of a component a_1 in presentation $a = a_1 + a_2$ where $a_1 \in \mathbf{Q}$, $a_1 = p^{-n} a'_1$, $a'_1 \in \mathbf{Z}$, and $a_2 \in \mathbf{Z}_p$. Any character χ of the type above has the form $x \rightarrow \chi_0(ax)$ for some rational a .

Let q be a quadratic form defined on the vector space V over F_v where v is one of the non-archimedean valuations of F . Suppose that L is a lattice in V such that $\chi(q(x)) = 1$ for any $x \in V$. The dual lattice $L^\#$ is defined as follows

$$(6) \quad L^\# = \{h \in V \mid \chi(\tilde{q}(x, h)) = 1 \quad \text{for} \quad \forall x \in V\}$$

where

$$(7) \quad \tilde{q}(x, y) = q(x + y) - q(x) - q(y)$$

is the bilinear form associated to q . Then the correspondence $q \mapsto \gamma_v^x(q)$ where

$$(8) \quad \gamma_v^x(q) = \frac{\sum_{h \in L^\# / L} \chi(q(h))}{\left| \sum_{h \in L^\# / L} \chi(q(h)) \right|}$$

defines a character of the Witt group $W(F_v)$ ([5]). (Over a field of zero characteristic we can identify the Witt group of quadratic forms with the Witt group of bilinear forms by the correspondence (7)). For an archimedean valuation v , the character γ_v^x is defined as follows.

$$(9) \quad \gamma_v^x(q) = \exp \frac{-\pi i \sigma(q)}{4}$$

if F_v is \mathbf{R} , and

$$(10) \quad \gamma_v^x(q) = 1$$

if F_v is \mathbf{C} . ($\sigma(q)$ denotes the signature of the quadratic form q). Now suppose that q is a quadratic form over the field F . Then q defines quadratic forms q_v over all F_v and the Weil quadratic reciprocity law asserts that

$$(11) \quad \prod_v \gamma_v^x(q_v) = 1$$

where v runs through all valuations of F .

If S is a symmetric bilinear form over \mathbf{Z} , on the lattice L such that $q(x) = S(x, x)$ is an even quadratic form, then applying (11) to $\varphi(x) = \frac{1}{2} q(x)$ and the character χ_0 of the ring $A_{\mathbf{Q}}$ defined above, one concludes that

$$(12) \quad e^{\frac{\pi i \sigma(q)}{4}} = \sum_{x \in L \not\equiv_p 0} e^{2\pi i \varphi(x)} \Big| \sum_{x \in L \not\equiv_p 0} e^{2\pi i \varphi(x)} \Big|$$

where L_p is the lattice of integer vectors in the p -adic completion of L . This is the essential part of Milgram's formula ([4]).

Now let us consider properties of the character γ^x in more detail.

Let F_p be one of the completions of F where p is a non-dyadic prime ideal.

LEMMA 1. Let q be a quadratic form over F_p . Let a be a unit in F_p . Denote by (aq) the quadratic form defined by $(aq)(x) = a \cdot q(x)$.

Then

$$(13) \quad \gamma_p^x(aq) = \left(\frac{a}{(\det \tilde{q}) \cdot s^{rkq}} \right) \gamma_p^x(q)$$

where $(-)$ is the quadratic residue symbol, s is the support of the character χ , and rkq is rank of the form q .

Remark. $\det \tilde{q}$ is defined up to a square in F_p , and therefore the quadratic residue symbol in (13) is well defined.

Now we consider dyadic valuations.

LEMMA 2. Let q denote a quadratic form over a ring of integers R_p of the dyadic field F_p such that the determinant of the associated form \tilde{q} (see (7)) is a unit in R_p . Let χ be a character of F_p with support R_p . If p is tamely ramified over \mathbf{Q} then

$$(14) \quad \text{Arf}(q \bmod p) = \gamma_p^x(2q).$$

Otherwise

$$(15) \quad \gamma_{\mathfrak{p}}^{\chi}(2q) = 1.$$

Remark. The condition on $\det \tilde{q}$ implies non-degeneracy of q at \mathfrak{p} .

§ 3. PROOF OF THE MAIN THEOREM

Note that the rank of q is even because determinant of the associated bilinear form is odd. Therefore

$$(16) \quad \gamma_v^{\chi}(aq) = \left(\frac{a}{(\det \tilde{q})} \right) \gamma_v^{\chi}(q)$$

for any character χ .

Now let us apply the Weil reciprocity law for the character χ with support in dyadic components equal to the integers in the corresponding ring, and to the forms q and $2q$.

We have

$$\prod_v \gamma_v^{\chi}(2q) = 1$$

$$\prod_v \gamma_v^{\chi}(q) = 1.$$

For an archimedean components we have $\gamma_v^{\chi}(2q) = \gamma_v^{\chi}(q)$ because both depend only on the signatures. Therefore dividing those two identities, and using lemma 2 and (16) we obtain the identity (4).

Remark. Levine's lemma which in a specialization of the theorem for $R = \mathbf{Z}$ in fact follows from Milgram's formula (12). We should not worry about ramification. Therefore lemma 1 can be used for the character χ_0 and is actually a classical property of Gauss sums ([2]). Lemma 2 in this case essentially contains in [1].

§ 4. PROOF OF THE LEMMAS

Proof of lemma 1. The Witt group of quadratic forms over a field of zero characteristic is generated by one-dimensional forms ([4]). Because γ^{χ} is a character of the Witt group it is enough to check the lemma for forms of one variable. Let π be a local parameter. Suppose that $q(x) = \alpha \pi^b x^2$,