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§ 2. WEIL-MILGRAM QUADRATIC RECIPROCITY LAW

Let  $A_F$  denote the adele group of the field  $F$  i.e. the group of infinite vectors  $(\dots x_v \dots)$  where  $v$  runs through all valuations of  $F$ ,  $x_v$  is an element of completion  $F_v$ , and all  $x_v$  except for finite number of  $v$ 's are integers.  $F$  is diagonally embedded in  $A_F$ . Let  $\chi$  denote a character of  $A_F$  trivial on  $F$ . Let  $\chi_0$  be the following special character of such type. On the archimedean component  $v$  of the adele  $\{x_v\} \in A_F$ ,  $\chi_0$  takes the value  $\exp(-2\pi i \text{Tr} x_v)$ , and on a non-archimedean component  $\exp(2\pi i \text{Tr} x_v)$ . Here  $\text{Tr}$  denotes the absolute trace from the  $v$ -component  $F_v$  of  $A_F$  to  $\mathbf{Q}_{\bar{v}}$  where  $\bar{v}$  is a valuation of  $\mathbf{Q}$  over which lies  $v$ . Recall that exponent of  $p$ -adic number  $a$  is defined as exponent of a component  $a_1$  in presentation  $a = a_1 + a_2$  where  $a_1 \in \mathbf{Q}$ ,  $a_1 = p^{-n} a'_1$ ,  $a'_1 \in \mathbf{Z}$ , and  $a_2 \in \mathbf{Z}_p$ . Any character  $\chi$  of the type above has the form  $x \rightarrow \chi_0(ax)$  for some rational  $a$ .

Let  $q$  be a quadratic form defined on the vector space  $V$  over  $F_v$  where  $v$  is one of the non-archimedean valuations of  $F$ . Suppose that  $L$  is a lattice in  $V$  such that  $\chi(q(x)) = 1$  for any  $x \in V$ . The dual lattice  $L^\#$  is defined as follows

$$(6) \quad L^\# = \{ h \in V \mid \chi(\tilde{q}(x, h)) = 1 \quad \text{for} \quad \forall x \in V \}$$

where

$$(7) \quad \tilde{q}(x, y) = q(x+y) - q(x) - q(y)$$

is the bilinear form associated to  $q$ . Then the correspondence  $q \mapsto \gamma_v^\chi(q)$  where

$$(8) \quad \gamma_v^\chi(q) = \sum_{h \in L^\# / L} \chi(q(h)) / \left| \sum_{h \in L^\# / L} \chi(q(h)) \right|$$

defines a character of the Witt group  $W(F_v)$  ([5]). (Over a field of zero characteristic we can identify the Witt group of quadratic forms with the Witt group of bilinear forms by the correspondence (7)). For an archimedean valuation  $v$ , the character  $\gamma_v^\chi$  is defined as follows.

$$(9) \quad \gamma_v^\chi(q) = \exp \frac{-\pi i \sigma(q)}{4}$$

if  $F_v$  is  $\mathbf{R}$ , and

$$(10) \quad \gamma_v^\chi(q) = 1$$

if  $F_v$  is  $\mathbf{C}$ . ( $\sigma(q)$  denotes the signature of the quadratic form  $q$ ). Now suppose that  $q$  is a quadratic form over the field  $F$ . Then  $q$  defines quadratic forms  $q_v$  over all  $F_v$  and the Weil quadratic reciprocity law asserts that

$$(11) \quad \prod_v \gamma_v^\chi(q_v) = 1$$

where  $v$  runs through all valuations of  $F$ .

If  $S$  is a symmetric bilinear form over  $\mathbf{Z}$ , on the lattice  $L$  such that  $q(x) = S(x, x)$  is an even quadratic form, then applying (11) to  $\varphi(x) = \frac{1}{2}q(x)$  and the character  $\chi_0$  of the ring  $A_{\mathbf{Q}}$  defined above, one concludes that

$$(12) \quad e^{\frac{\pi i \sigma(q)}{4}} = \left| \sum_{x \in L / \frac{L}{p}} e^{2\pi i \varphi(x)} \right| \left| \sum_{x \in L / \frac{L}{p}} e^{2\pi i \varphi(x)} \right|$$

where  $L_p$  is the lattice of integer vectors in the  $p$ -adic completion of  $L$ . This is the essential part of Milgram's formula ([4]).

Now let us consider properties of the character  $\gamma^\chi$  in more detail.

Let  $F_{\mathfrak{p}}$  be one of the completions of  $F$  where  $\mathfrak{p}$  is a non-dyadic prime ideal.

LEMMA 1. Let  $q$  be a quadratic form over  $F_{\mathfrak{p}}$ . Let  $a$  be a unit in  $F_{\mathfrak{p}}$ . Denote by  $(aq)$  the quadratic form defined by  $(aq)(x) = a \cdot q(x)$ . Then

$$(13) \quad \gamma_{\mathfrak{p}}^\chi(aq) = \left( \frac{a}{(\det \tilde{q}) \cdot \mathfrak{s}^{rkq}} \right) \gamma_{\mathfrak{p}}^\chi(q)$$

where  $(-)$  is the quadratic residue symbol,  $\mathfrak{s}$  is the support of the character  $\chi$ , and  $rkq$  is rank of the form  $q$ .

*Remark.*  $\det \tilde{q}$  is defined up to a square in  $F_{\mathfrak{p}}$ , and therefore the quadratic residue symbol in (13) is well defined.

Now we consider dyadic valuations.

LEMMA 2. Let  $q$  denote a quadratic form over a ring of integers  $R_{\mathfrak{p}}$  of the dyadic field  $F_{\mathfrak{p}}$  such that the determinant of the associated form  $\tilde{q}$  (see (7)) is a unit in  $R_{\mathfrak{p}}$ . Let  $\chi$  be a character of  $F_{\mathfrak{p}}$  with support  $R_{\mathfrak{p}}$ . If  $\mathfrak{p}$  is tamely ramified over  $\mathbf{Q}$  then

$$(14) \quad \text{Arf}(q \bmod \mathfrak{p}) = \gamma_{\mathfrak{p}}^\chi(2q).$$

Otherwise

$$(15) \quad \gamma_p^\chi(2q) = 1.$$

*Remark.* The condition on  $\det \tilde{q}$  implies non-degeneracy of  $q$  at  $p$ .

### § 3. PROOF OF THE MAIN THEOREM

Note that the rank of  $q$  is even because determinant of the associated bilinear form is odd. Therefore

$$(16) \quad \gamma_v^\chi(aq) = \left( \frac{a}{(\det \tilde{q})} \right) \gamma_v^\chi(q)$$

for any character  $\chi$ .

Now let us apply the Weil reciprocity law for the character  $\chi$  with support in dyadic components equal to the integers in the corresponding ring, and to the forms  $q$  and  $2q$ .

We have

$$\prod_v \gamma_v^\chi(2q) = 1$$

$$\prod_v \gamma_v^\chi(q) = 1.$$

For an archimedean components we have  $\gamma_v^\chi(2q) = \gamma_v^\chi(q)$  because both depend only on the signatures. Therefore dividing those two identities, and using lemma 2 and (16) we obtain the identity (4).

*Remark.* Levine's lemma which in a specialization of the theorem for  $R = \mathbf{Z}$  in fact follows from Milgram's formula (12). We should not worry about ramification. Therefore lemma 1 can be used for the character  $\chi_0$  and is actually a classical property of Gauss sums ([2]). Lemma 2 in this case essentially contains in [1].

### § 4. PROOF OF THE LEMMAS

*Proof of lemma 1.* The Witt group of quadratic forms over a field of zero characteristic is generated by one-dimensional forms ([4]). Because  $\gamma^\chi$  is a character of the Witt group it is enough to check the lemma for forms of one variable. Let  $\pi$  be a local parameter. Suppose that  $q(x) = \alpha \pi^b x^2$ ,