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LEVINE'S FORMULA IN KNOT THEORY AND QUADRATIC RECIPROCITY LAW

by A. LIBGOBER

§ 1. INTRODUCTION

A k -knot is a k -dimensional submanifold of S^{k+2} which is homeomorphic to a sphere. Any knot K is bounded by a submanifold $F^{k+1} \subset S^{k+2}$ which is called the Seifert surface of K . One associates with K the Alexander polynomial $\Delta(t)$. Moreover if $k = 4n + 1$ then one may associate with F^{4n+2} the non-degenerate quadratic Z_2 -form φ on $H_{2n+1}(F^{4n+2}, Z_2)$. Levine's formula asserts that the Arf invariant of this quadratic form is trivial if $\Delta(-1) \equiv \pm 1 \pmod{8}$ and is non trivial if $\Delta(-1) \equiv \pm 3 \pmod{8}$.

Levine's proof consists of two parts. The first one is topological and states that both the quadratic function φ on $H_{2n+1}(F^{4n+2}, Z_2)$ which is used for the computation of the Arf invariant, and the Alexander polynomial can be expressed in terms of the Seifert pairing L of F^{4n+2} , which is the bilinear form on $H_{2n+1}(F^{4n+2}, Z)$. Namely

$$(1) \quad \varphi(x \bmod 2) \equiv L(x, x) \bmod 2$$

and

$$(2) \quad \Delta(t) = \det(L - tL^t)$$

i.e.

$$(2a) \quad \Delta(-1) = \det(L + L^t).$$

($L^t(x, y)$ is by definition $L(y, x)$).

The second part of Levine's proof is the remarkable observation that the Arf invariant of a quadratic function defined by (1) can be found in terms of the associated bilinear form $L + L^t$. He proved the following (cf. [6])

Levine's lemma. Let $L(x, y)$ denote a bilinear form on a free abelian group such that $d = \det(L + L^t)$ is odd. Let φ denote the quadratic function on $V \otimes \mathbb{Z}_2$ defined by (1). Then

$$\text{Arf } \varphi = \begin{cases} 1 & \text{if } d \equiv \pm 1 \pmod{8} \\ -1 & \text{if } d \equiv \pm 3 \pmod{8} \end{cases}$$

(We suppose that the range of Arf invariant is ± 1).

The purpose of this paper is to show that Levine's lemma is closely related to the Weil-Milgram reciprocity law ([4], [5]). In fact our main result is a generalization of Levine's lemma to arbitrary algebraic number fields.

Let F denote an algebraic number field and L be a bilinear form on a projective module P over the ring R of integers in F . Suppose that the determinant of the symmetrized form $d = \det(L + L^t)$ is relatively prime to 2.

For any dyadic (i.e. dividing 2) prime ideal $\mathfrak{p}$, let $\text{Arf } L_{\mathfrak{p}}$ denote the Arf invariant of the quadratic form $x \rightarrow L(x, x) \pmod{\mathfrak{p}}$ over $P \otimes R/\mathfrak{p}$. For $a \in R$ and a nondyadic prime ideal $\mathfrak{p}$ we denote the quadratic residue symbol by

$$\left(\frac{a}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } a \text{ is square in } R/\mathfrak{p} \\ 0 & \text{if } a \in \mathfrak{p} \\ -1 & \text{otherwise.} \end{cases}$$

In the same way we denote the multiplicative extension of $\left(\frac{a}{\mathfrak{p}}\right)$ on the group of all non-dyadic ideals of R .

THEOREM. *With the above notations,*

$$(4) \quad \prod_{\mathfrak{p}} \text{Arf } L_{\mathfrak{p}} = \left(\frac{2}{dR}\right)$$

where $\mathfrak{p}$ runs through all tamely ramified dyadic prime ideals of R and dR is the principal ideal generated by d .

In § 2 we give the necessary definitions and formulate two lemmas about Gauss sums for bilinear forms.

In § 3 we prove the theorem, using the results of § 2. The proofs of the lemmas in § 2 is given in § 4. Finally I would like to thank A. Adler, W. Pardon, and C. Weibel for useful discussions.