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LEVINE'S FORMULA IN KNOT THEORY AND QUADRATIC RECIPOCITY LAW

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§ 1. Introduction

A k-knot is a k-dimensional submanifold of S^{k+2} which is homeomorphic to a sphere. Any knot K is bounded by a submanifold $F^{k+1} \subset S^{k+2}$ which is called the Seifert surface of K. One associates with K the Alexander polynomial $\Delta(t)$. Moreover if $k = 4n + 1$ then one may associate with F^{4n+2} the non-degenerate quadratic Z_2-form φ on $H_{2n+1}(F^{4n+2}, Z_2)$. Levine's formula asserts that the Arf invariant of this quadratic form is trivial if $\Delta(-1) \equiv \pm 1 \pmod 8$ and is non trivial if $\Delta(-1) \equiv \pm 3 \pmod 8$.

Levine's proof consists of two parts. The first one is topological and states that both the quadratic function φ on $H_{2n+1}(F^{4n+2}, Z_2)$ which is used for the computation of the Arf invariant, and the Alexander polynomial can be expressed in terms of the Seifert pairing L of F^{4n+2}, which is the bilinear form on $H_{2n+1}(F^{4n+2}, Z)$. Namely

(1) $$\varphi(x \bmod 2) \equiv L(x, x) \bmod 2$$

and

(2) $$\Delta(t) = \det(L - tL^t)$$

i.e.

(2a) $$\Delta(-1) = \det(L + L^t).$$

($L^t(x, y)$ is by definition $L(y, x)$).

The second part of Levine's proof is the remarkable observation that the Arf invariant of a quadratic function defined by (1) can be found in terms of the associated bilinear form $L + L^t$. He proved the following (cf. [6])

Levine's lemma. Let $L(x, y)$ denote a bilinear form on a free abelian group such that $d = \det(L+L^t)$ is odd. Let φ denote the quadratic function on $V \otimes Z_2$ defined by (1). Then

$$\operatorname{Arf} \varphi = \begin{cases} 1 & \text{if} \quad d \equiv \pm 1 \pmod 8 \\ -1 & \text{if} \quad d \equiv \pm 3 \pmod 8 \end{cases}$$

(We suppose that the range of Arf invariant is ± 1).

The purpose of this paper is to show that Levine's lemma is closely related to the Weil-Milgram reciprocity law ([4], [5]). In fact our main result is a generalization of Levine's lemma to arbitrary algebraic number fields.

Let F denote an algebraic number field and L be a bilinear form on a projective module P over the ring R of integers in F. Suppose that the determinant of the symmetrized form $d = \det(L+L^t)$ is relatively prime to 2.

For any dyadic (i.e. dividing 2) prime ideal $\mathfrak{p}$, let $\operatorname{Arf} L_p$ denote the Arf invariant of the quadratic form $x \to L(x, x) \bmod \mathfrak{p}$ over $P \otimes R/\mathfrak{p}$. For $a \in R$ and a nondyadic prime ideal $\mathfrak{p}$ we denote the quadratic residue symbol by

$$\left(\frac{a}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if} \quad a \text{ is square in } R/\mathfrak{p} \\ 0 & \text{if} \quad a \in \mathfrak{p} \\ -1 & \quad \text{otherwise}. \end{cases}$$

In the same way we denote the multiplicative extension of $\left(\frac{a}{\mathfrak{p}}\right)$ on the group of all non-dyadic ideals of R.

THEOREM. *With the above notations,*

$$\prod_{\mathfrak{p}} \operatorname{Arf} L_{\mathfrak{p}} = \left(\frac{2}{dR}\right) \tag{4}$$

where $\mathfrak{p}$ runs through all tamely ramified dyadic prime ideals of R and dR is the principal ideal generated by d.

In § 2 we give the necessary definitions and formulate two lemmas about Gauss sums for bilinear forms.

In § 3 we prove the theorem, using the results of § 2. The proofs of the lemmas in § 2 is given in § 4. Finally I would like to thank A. Adler, W. Pardon, and C. Weibel for useful discussions.