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The equivalence of Characterizations C2 and C8 is proved in [Wagreich 2, p. 66; Saito 2, Theorem 1.9]. In fact, the zero loci of the germs  $\tilde{E}_6$ ,  $\tilde{E}_7$  and  $\tilde{E}_8$  have minimal resolution as above with  $E^2 = -3$ ,  $-2$  and  $-1$  respectively.

*Characterization C9.* The germ  $f^{-1}(0)$  is isomorphic to  $g^{-1}(0)$ , where  $g$  is a weighted homogeneous polynomial with weights  $w_i$  satisfying  $w_0^{-1} + \dots + w_n^{-1} = n/2$ .

The equivalence of Characterizations C2 and C9 is proved in [Saito 2, Satz 2.11]. In fact, the germs in Table 2b have the following weights:

GERM	WEIGHTS
$P_8$	(3, 3, 3)
$X_9$	(4, 4)
$J_{10}$	(3, 6)

## APPENDIX II

### THE MONODROMY GROUPS OF THE MINIMAL HYPERBOLIC GERMS

**PROPOSITION.** *The monodromy groups of the germs  $P_9$ ,  $X_{10}$ , and  $J_{11}$  have exponential growth.*

In this appendix, we present an (unpublished) proof of this proposition due to E. Looijenga. In fact, we will show that these groups have  $PSL(2, \mathbf{Z})$  as subquotient (quotient of a subgroup). We let  $O(V)$  denote the orthogonal group of a  $\mathbf{Z}$ - or  $\mathbf{R}$ -module  $V$  equipped with a bilinear form.

Suppose  $G$  is a polyhedral graph whose edges are weighed by non-zero integers. By convention, the weight 1 is omitted. Let  $L_G$  denote the free  $\mathbf{Z}$ -module generated by the vertices  $v_1, \dots, v_n$  of  $G$ . Define a symmetric bilinear form  $( , )$  on  $L_G$  by setting  $(v_i, v_i) = -2$ , and  $(v_i, v_j) = 0$  if there is no edge from  $v_i$  to  $v_j$ , otherwise equal to the weight on this edge. Conversely, given a symmetric integral bilinear form  $( , )$  on a free module  $L$  with basis  $\alpha_1, \dots, \alpha_n$  with the property that  $(\alpha_i, \alpha_i) = -2$  for all  $i$ , one associates a graph to it in the obvious way.

For  $\alpha \in L_G$ , let  $s_\alpha$  (*reflection in*  $\alpha$ ) be the isometry of  $L_G$  defined by

$$s_\alpha(\beta) = \beta - 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \alpha$$

for  $\beta \in L_G$ . The *reflection group*  $\mathcal{R}(G)$  of the graph  $G$  is defined to be the subgroup of  $O(L_G)$  generated by  $s_{\alpha_1}, \dots, s_{\alpha_n}$ .

*Example 1.* Consider a reduced irreducible root system in a vector space  $V$ . Let  $\alpha_1, \dots, \alpha_n$  be a collection of simple roots, let  $L$  be the free  $\mathbf{Z}$ -module spanned by the  $\alpha_i$ , and let  $(, )$  be the negative of an invariant bilinear form [Serre, Chapter 5]. If  $(\alpha_i, \alpha_i) = -2$  for all  $i$ , then the corresponding graph must be of type  $A_k, D_k, E_6, E_7$  or  $E_8$ . The reflection group of these graphs equals the Weyl group, the group generated by reflections in all the roots [Serre, p. V-16]. Furthermore, the reflection group together with the generators  $s_{\alpha_1}, \dots, s_{\alpha_n}$  forms a Coxeter system [Bourbaki, p. 92]. (A *Coxeter system* is a group  $G$ , a collection of elements  $g_1, \dots, g_n$  and a symmetric integral  $n \times n$  matrix  $\{m_{ij}\}$  with  $m_{ii} = 1$  and  $2 \leq m_{ij} \leq \infty$  for  $i \neq j$ , with the property that  $G$  is isomorphic to the free group with generators  $g_1, \dots, g_n$  and relations  $(g_i, g_j)^{m_{ij}} = 1$ , for all  $i, j$ .)

*Example 2.* The monodromy group of a germ  $f$  is the reflection group of a quadratic form diagram for  $f$  (Sections 13 and 14). When this diagram is a tree (which is only possible for the simple germs), its reflection group together with the generators  $T_1, \dots, T_\mu$  forms a Coxeter system. In general, this reflection group is not a Coxeter system [A'Campo 2, II, p. 403].

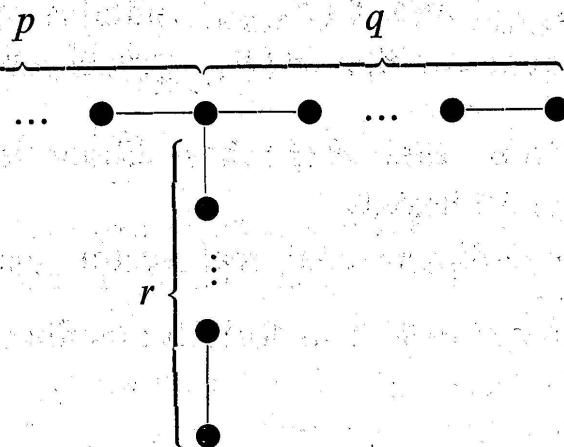
*Lemma* [Gabrielov 3]. If the graph  $G'$  is a subgraph of the graph  $G$ , then  $\mathcal{R}(G')$  is a subquotient of  $\mathcal{R}(G)$ .

*Proof.* Let  $\alpha'_1, \dots, \alpha'_m$  be a basis of  $L_{G'}$  corresponding to the vertices of  $G'$ , let  $\alpha_1, \dots, \alpha_m$  be the corresponding elements in  $L_G$ , and extend this to a basis  $\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n$  of  $L_G$  corresponding to the vertices of  $G$ . The map  $\alpha'_i \rightarrow \alpha_i$  is an isometric embedding of  $L_{G'}$  in  $L_G$ . Let  $\mathcal{R}'$  be the subgroup of  $\mathcal{R}(G)$  generated by  $s_{\alpha_1}, \dots, s_{\alpha_m}$ ; it has a presentation with these generators and certain relations. Any relation among these  $s_{\alpha_i}$  holds also for  $s_{\alpha_i}|_{L_{G'}} = s_{\alpha'_i}$ . Thus  $\mathcal{R}'$  maps onto  $\mathcal{R}(G')$ .

*Fact.* If a subquotient of a group  $G$  has exponential growth, then so does  $G$ .

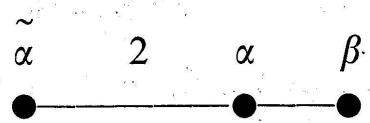
*Proof of Proposition.* 1. A quadratic form diagram for the germs  $P_9, X_{10}$ , and  $J_{11}$  is given in column 5 of Table 2. These graphs contain a

subgraph of the form  $T_{3,3,4}$ ,  $T_{2,4,5}$ , and  $T_{2,3,7}$  respectively, where  $T_{p,q,r}$  is the graph

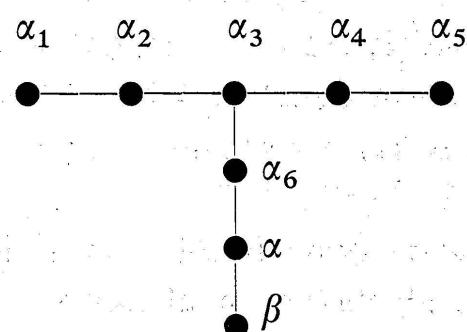


Hence it suffices to show that the reflection groups of these graphs have exponential growth.

Let  $\Gamma$  be the graph



with vertices corresponding to basis elements  $\tilde{\alpha}, \alpha, \beta$  in  $L_\Gamma$  as indicated. We claim that  $\mathcal{R}(T_{p,q,r})$  has  $\mathcal{R}(\Gamma)$  as subquotient, for  $(p, q, r) = (3, 3, 4)$ ,  $(2, 4, 5)$ , and  $(2, 3, 7)$ . Consider (for example)  $T_{3,3,4}$ , with vertices corresponding to basis elements  $\alpha_i \in L_{T_{3,3,4}}$  as indicated:



This contains the graph  $E_6$ . Let

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \in L_{T_{3,3,4}}$$

be the largest root of  $E_6$  [Bourbaki, p. 165]. Since all the roots of  $E_6$  are the same length,  $(\tilde{\alpha}, \tilde{\alpha}) = -2$ . The lattice spanned by  $\tilde{\alpha}, \alpha, \beta$  has diagram  $\Gamma$ . The reflections  $s_\alpha$  and  $s_\beta$  are in  $\mathcal{R}(T_{3,3,4})$ . We claim that  $s_{\tilde{\alpha}}$  is in  $\mathcal{R}(T_{3,3,4})$  as well: The restriction  $s_{\tilde{\alpha}}|_{L_{E_6}}$  is in  $\mathcal{R}(E_6)$ , since  $E_6$  is a root system. Hence  $s_{\tilde{\alpha}}|_{L_{E_6}} = (s_{\alpha_{i(1)}} \circ \dots \circ s_{\alpha_{i(m)}})|_{L_{E_6}}$  for some  $1 \leq i(1), \dots, i(m) \leq 6$ .

Also,  $s_{\alpha}^{\sim}$  and  $s_{\alpha_i(1)} \circ \dots \circ s_{\alpha_i(m)}$  are both the identity when restricted to the orthogonal complement of  $L_{E_6} \otimes \mathbf{R}$  in  $L_{T_{3,3,4}} \otimes \mathbf{R}$ . Thus  $s_{\alpha}^{\sim} = s_{\alpha_i(1)} \circ \dots \circ s_{\alpha_i(m)}$ , and  $\mathcal{R}(T_{3,3,4})$  contains  $s_{\alpha}^{\sim}$ . A proof similar to that of the lemma then shows that  $\mathcal{R}(T_{3,3,4})$  has subquotient  $\mathcal{R}(\Gamma)$ .

3. Next we show that  $\mathcal{R}(\Gamma)$  has subquotient  $PSL(2, \mathbf{Z})$ . This uses [E. Artin, Chapter V] heavily.

Let  $V$  be the 3-dimensional real vector space  $L_{\Gamma} \otimes \mathbf{R}$ . The bilinear form  $( , )$  of  $\Gamma$  extends to  $V$ . This form is indefinite since  $\tilde{\alpha} + \alpha$  has length 0. Let

$$0'(V) = \{f \in 0(V) : \det f = 1 \text{ and spinor norm of } f \text{ equal to } 1 \mathbf{R}^{*2}\}.$$

Since  $V$  is indefinite, it is known [E. Artin, p. 200] that

$$(1) \quad 0'(V) \xrightarrow{\cong} PSL(2, \mathbf{R}).$$

Since  $PSL(2, \mathbf{R})$  contains  $PSL(2, \mathbf{Z})$  as a subgroup, the idea is to find elements of  $\mathcal{R}(\Gamma) \subset 0(\Gamma)$  which are in  $0'(V)$  and map to generators of  $PSL(2, \mathbf{Z})$ . The standard generators of  $PSL(2, \mathbf{Z})$  are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with relations  $S^2 = (ST)^3 = 1$ . By inspection, it is found that the elements  $s_{\alpha}^{\sim} s_{\beta}$  and  $s_{\beta} s_{\alpha}$  of  $0(L)$  satisfy  $(s_{\alpha}^{\sim} s_{\beta})^2 = (s_{\beta} s_{\alpha})^3 = 1$ , and have determinant equal to 1 and spinor norm equal to  $1 \mathbf{R}^{*2}$ . Therefore we would like to choose the isomorphism (1) such that  $s_{\alpha}^{\sim} s_{\beta}$  maps to  $S$ , and  $s_{\alpha}^{\sim} s_{\alpha} = (s_{\alpha}^{\sim} s_{\beta})^{-1} (s_{\beta} s_{\alpha})$  maps to  $S^{-1} (ST) = T$ .

The isomorphism (1) is done in two steps. First, let  $D_0(V)$  be the elements of the Clifford algebra of  $V$  of norm 1; then [E. Artin, p. 199]

$$(2) \quad D_0(V)/\{\pm 1\} \xrightarrow{\cong} 0'(V).$$

We do not need to know exactly what this isomorphism is, but only that

$$v \circ w \rightarrow s_v s_w$$

for elements  $v, w$  in  $V$  regarded as a subspace of the Clifford algebra, and  $v \circ w$  their product. Hence under the above isomorphisms

$$(3) \quad \frac{1}{2} \tilde{\alpha} \circ \beta \mapsto s_{\alpha}^{\sim} s_{\beta}, \quad \frac{1}{2} \tilde{\alpha} \circ \alpha \mapsto s_{\alpha}^{\sim} s_{\alpha}.$$

Secondly [E. Artin, p. 199],

$$(4) \quad D_0(V) / \{\pm 1\} \cong PSL(2, \mathbf{R}).$$

We examine this more closely. Let

$$A_1 = \sqrt{2}(\alpha + \tilde{\alpha} + \beta/2), \quad A_2 = \tilde{\alpha}/\sqrt{2}, \quad A_3 = \beta/\sqrt{2}.$$

Then  $A_1, A_2, A_3$  is an orthogonal basis of  $V$ , and the matrix of  $(,)$  with respect to this basis is the diagonal matrix  $\langle +1, -1, -1 \rangle$ . Let  $C^+(V)$  be the subspace of the Clifford algebra of  $V$  spanned by the elements of even grading;  $C^+(V)$  is generated by  $1, i_1, i_2, i_3$ , where  $i_1 = A_2 \circ A_3$ ,  $i_2 = A_3 \circ A_1$ , and  $i_3 = A_1 \circ A_2$ , and has multiplication table as in [E. Artin, top of p. 200] with  $a = -1$ . The map

$$C^+(V) \rightarrow M(2, \mathbf{R})$$

(where  $M(2, \mathbf{R})$  is the algebra of all  $2 \times 2$  matrices over  $\mathbf{R}$ ) defined by

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & i_1 &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ i_2 &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & i_3 &\mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

is an isomorphism. (This is slightly different from the isomorphism of [E. Artin, p. 200].), and the restriction of this map to  $D_0(V)$  gives the isomorphism (4). Furthermore,

$$(5) \quad \frac{1}{2} \tilde{\alpha} \circ \beta = i_1 \mapsto S, \quad \frac{1}{2} \tilde{\alpha} \circ \alpha = 1 - \frac{1}{2}(i_1 + i_3) \mapsto T$$

under this isomorphism. Combining isomorphisms (2) and (4) gives isomorphism (1), and (3) and (5) show that

$$s_{\tilde{\alpha}} s_{\beta} \mapsto S, \quad s_{\tilde{\alpha}} s_{\alpha} \mapsto T$$

under isomorphism (1). Thus  $\mathcal{R}(\Gamma)$  maps onto  $PSL(2, \mathbf{Z})$ , and hence has  $PSL(2, \mathbf{Z})$  as subquotient.

4. Finally,  $PSL(2, \mathbf{Z})$  is isomorphic to the free product  $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z})$  [Serre, ch. 7; Lehner, p. 59], which has exponential growth.

TABLE 1 RATIONAL DOUBLE POINTS

(1) $f(x, y, z)$	(2) $(p, q, r)$	(3) Dual Graph of Resolution	(4) Name	(5) Subgroup $G$ of $S^3$	(6) Order of $G$	(7) Weights
$x^{k+1} + y^2 + z^2$	(1, 1, $k$ )		$A_k$ $k \geq 1$	cyclic	$k+1$	$(k+1, 2, 2)$
$x^{k-1} + xy^2 + z^2$	(2, 2, $k-2$ )		$D_k$ $k \geq 4$	binary dihedral	$4(k-2)$	$(k-1, 2(k-1)/(k-2), 2)$
$x^3 + y^4 + z^2$	(2, 3, 3)		$E_6$	binary tetrahedral	24	(3, 4, 2)
$x^3 + xy^3 + z^2$	(2, 3, 4)		$E_7$	binary octahedral	48	(3, 9/2, 2)
$x^3 + y^5 + z^2$	(2, 3, 5)		$E_8$	binary icosahedral	120	(3, 5, 2)

Column 1 lists the germ  $f(x, y, z)$ . Column 3 lists the dual graph of the minimal resolution of  $f^{-1}(0)$ . The name of the graph is given in column 4. Each graph is of type  $T_{p,q,r}$ , where  $(p, q, r)$  are listed in column 2. Every vertex of the graph represents a nonsingular rational curve of self-intersection -2. The analytic set  $f^{-1}(0)$  is isomorphic to  $\mathbb{C}^2/G$ , where  $G$  is the finite subgroup of  $S^3$  listed in column 5. Each germ  $f$  is weighted homogeneous, with weights as listed in column 7.

TABLE 2 GERMS OF LOW MINOR NUMBER

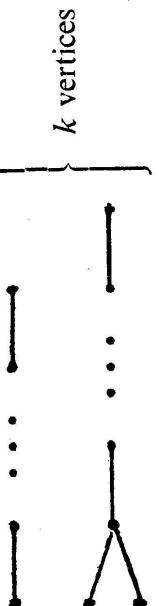
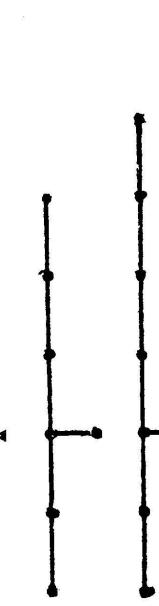
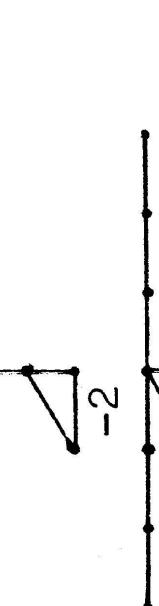
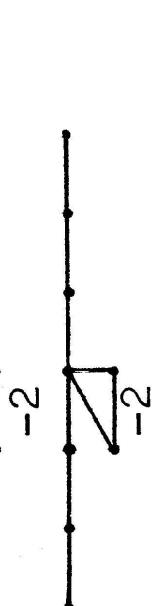
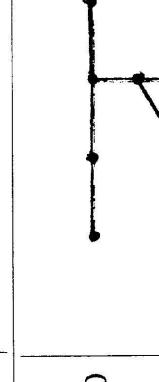
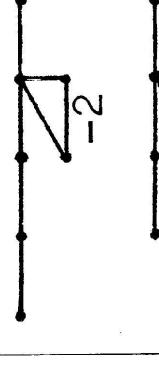
(1)	(2) Name	(3) $\mu$	(4) $f(z_0, \dots, z_n)$	(5) Quadratic form diagram
a (simple germs)	$A_k, k \geq 1$	$k$	$z_0^{k+1}$	
	$D_k, k \geq 4$	$k$	$z_0^{k+1} + z_0 z_1^2$	
	$E_6$	6	$z_0^3 + z_1^4$	
	$E_7$	7	$z_0^3 + z_0 z_1^3$	
	$E_8$	8	$z_0^3 + z_1^5$	
b (almost- simple germs)	$P_8$ or $\tilde{E}_6$	8	$z_0^3 + z_1^2 z_2 + a z_0 z_2^2 + b z_2^3, 4a^3 + 27b^2 \neq 0$	
	$X_9$ or $\tilde{E}_7$	9	$z_0 z_1 (z_0 - z_1) (z_0 - az_1), a \neq 0, 1$	
	$J_{10}$ or $\tilde{E}_8$	10	$z_0 (z_0 - z_1^2) (z_0 - az_1^2), a \neq 0, 1$	

TABLE 2 (*continuation*)

$c$	$P_9$ or $T_{3,3,4}$	9	$z_0^2 z_2 + z_1^3 + z_1^2 z_2 + az_2^4, a \neq 0$	
(minimal hyperbolic germs)	$X_{10}$ or $T_{2,4,5}$	10	$z_0^4 + z_0^2 z_1^2 + az_1^5, a \neq 0$	
	$J_{11}$ or $T_{2,3,7}$	11	$z_0^3 + \varepsilon z_0^2 z_1^2 + az_1^7, a \neq 0, \varepsilon = \pm 1$	

Table 2 is divided into parts a, b, and c as indicated in column 1. These parts are referred to in the text as Table 2a, etc. Column 4 contains the equation of the germ; to each equation must be added the quadratic form  $z_r^2 + \dots + z_n^2$  in the variables  $z_r, \dots, z_n$  not occurring in the equation. Column 2 gives the type of the germ in Arnold's notation, column 3 gives its Milnor number  $\mu$ , and column 5 gives a quadratic form diagram.

TABLE 3  
SIMPLIFICATION TABLE

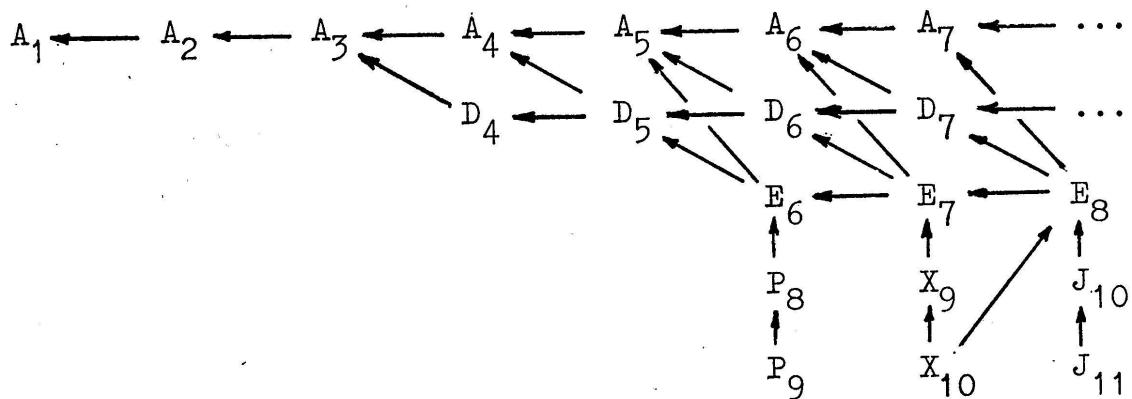


Table 3 lists some (but not all) of the simplifications that occur among the germs of Table 2.

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