

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	25 (1979)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	FIFTEEN CHARACTERIZATIONS OF RATIONAL DOUBLE POINTS AND SIMPLE CRITICAL POINTS
Autor:	Durfee, Alan H.
Kapitel:	7. Volume
DOI:	https://doi.org/10.5169/seals-50375

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

PROPOSITION 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of V is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

Characterization A6. The local fundamental group of $f^{-1}(0)$ is finite. Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of V from a resolution [Mumford], and singularities V with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When V is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

7. VOLUME

Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f . There is an $\varepsilon > 0$ such that $f^{-1}(\mathbf{0})$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leq \varepsilon$. (See Section 12.) For $t \in \mathbf{C}$, let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where D_ε^6 is the closed disk of radius ε about $\mathbf{0}$. The function $f(x, y, z)$ takes the constant value t on V_t , so $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$ there. Hence a nowhere-vanishing holomorphic two-form ω_t on V_t may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

Characterization A7. The integral $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$ is finite.

Note that the form $\omega_0 \wedge \bar{\omega}_0$ takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let $\Delta(r) = \{t \in \mathbf{C}: t < r\}$, let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let $\text{vol}(X(r))$ be its volume in \mathbf{C}^3 .

Characterization A7'. $\lim_{r \rightarrow 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of \mathbf{C}^3 . Characterizations A7 and A7' are equivalent since

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{1}{r^2} \operatorname{vol}(X(r)) &= \lim_{r \rightarrow 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega} \\ &= \lim_{r \rightarrow 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \bar{dt}, \end{aligned}$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2} \right) dt \wedge \bar{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0.$$

B. NINE CHARACTERIZATIONS OF SIMPLE CRITICAL POINTS

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \dots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

THEOREM B. *Let $f(z_0, \dots, z_n)$ with $n \geq 1$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that $\mathbf{0}$ is an isolated critical point of f . Then Characterizations B1 through B9 are equivalent.*

8. THE CLASSIFICATION OF RIGHT EQUIVALENCE CLASSES

Let \mathcal{O} be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on \mathbf{C}^{n+1} . (In other words, \mathcal{O} is just the ring $\mathbf{C}\{z_0, \dots, z_n\}$ of convergent power series.) The ring \mathcal{O} is local with maximal ideal

$$m = \{f \in \mathcal{O} : f(\mathbf{0}) = 0\}.$$