

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 25 (1979)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FIFTEEN CHARACTERIZATIONS OF RATIONAL DOUBLE POINTS AND SIMPLE CRITICAL POINTS  
**Autor:** Durfee, Alan H.  
**Kapitel:** 5. Quotient singularities  
**DOI:** <https://doi.org/10.5169/seals-50375>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 21.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## 5. QUOTIENT SINGULARITIES

Let  $U$  be a neighborhood of the origin  $0$  in  $\mathbb{C}^2$  and let  $H$  be a finite group of analytic automorphisms of  $U$  fixing  $0$ . The quotient space  $U/H$  has the structure of a normal two-dimensional complex analytic space with an isolated singularity, and the projection map  $U \rightarrow U/H$  is analytic [Cartan]. An analytic space  $V$  is called a *quotient singularity* if there is a  $U$  and  $H$  as above such that  $V$  is isomorphic to  $U/H$ .

An important example of a quotient singularity is  $\mathbb{C}^2/G$ , where  $G$  is some finite subgroup of  $GL(2, \mathbb{C})$ . The space  $\mathbb{C}^2/G$  is not just analytic, but algebraic. For any finite subgroup  $G$  of  $GL(2, \mathbb{C})$ , the ring of functions on the algebraic variety  $\mathbb{C}^2/G$  is isomorphic to the subring of invariant polynomials in  $GL(2, \mathbb{C})$ . Hence to find  $\mathbb{C}^2/G$  it suffices to find this subring of invariant polynomials. Note that a finite subgroup  $G$  of  $GL(2, \mathbb{C})$  or  $SL(2, \mathbb{C})$  is conjugate to a finite subgroup of  $U(2)$  or  $SU(2)$  respectively, since it is possible to choose an invariant Hermitian metric on  $\mathbb{C}^2$ . A subgroup  $G \subset GL(2, \mathbb{C})$  is *small* if no  $g \in G$  has 1 as an eigenvalue of multiplicity one. [Prill, p. 380].

**PROPOSITION 5.1.** *Let  $V$  be the germ of a normal two-dimensional complex analytic space. The following statements are equivalent.*

- (a)  $V$  is a quotient singularity.
- (b)  $V$  is isomorphic to  $\mathbb{C}^2/G$ , for some finite subgroup  $G$  of  $GL(2, \mathbb{C})$ .
- (c)  $V$  is isomorphic to  $\mathbb{C}^2/G$ , for some small finite subgroup of  $GL(2, \mathbb{C})$ .

Condition (a) implies condition (b) by the usual linearization argument [Brieskorn 2, Lemma 2.2]. It is shown in [Prill, p. 380] that condition (b) implies condition (c). Obviously (c) implies (a). The following theorem is also proved in [Prill]: Let  $G$  and  $G'$  be small finite subgroups of  $GL(2, \mathbb{C})$ . Then the analytic spaces  $\mathbb{C}^2/G$  and  $\mathbb{C}^2/G'$  are isomorphic if and only if  $G$  and  $G'$  are conjugate.

**Characterization A5.** The analytic space  $f^{-1}(0)$  is a quotient singularity.

Since quotient singularities are rational [Brieskorn 2, p. 340], Characterization A5 implies Characterization A2. The converse will follow in round-about fashion.

Consider  $SU(2)$ , which is of course isomorphic to the group  $S^3$  of unit quaternions. The finite subgroups of  $S^3$  are the cyclic group and the inverse

images of the finite subgroups of the rotation group  $SO(3)$  under the double cover  $S^3 \rightarrow SO(3)$ ; these groups are listed in column 5 of Table 1.

**PROPOSITION 5.2.** *Let  $G$  be a non-trivial finite subgroup of  $SU(2)$  as listed in column 5 of Table 1. Then  $\mathbb{C}^2/G$  is isomorphic to  $f^{-1}(0)$ , where  $f$  is the corresponding polynomial in column 1.*

In particular, for each polynomial  $f$  in column 1 of Table 1 the analytic space  $f^{-1}(0)$  is isomorphic to a quotient singularity. This proposition is proved by classical invariant theory. For the cyclic group it is easy: Let  $G \subset SU(2)$  be the cyclic group of order  $k$ , generated by the transformation  $(u, v) \rightarrow (\eta u, \eta^{-1}v)$  where  $\eta$  is a primitive  $k$ -th root of unity. Then we claim that  $\mathbb{C}^2/G$  is isomorphic to

$$V = \{ (x, y, z) \in \mathbb{C}^3 : x^k = yz \}.$$

Let  $p_1(u, v) = uv$ ,  $p_2(u, v) = u^k$ ,  $p_3(u, v) = v^k$ , and let  $p = (p_1, p_2, p_3)$  define a map of  $\mathbb{C}^2$  to  $\mathbb{C}^3$ . The image of  $p$  is exactly  $V$ . Since  $p_i(gu, gv) = p_i(u, v)$  for all  $g$  in  $G$ , the map  $p$  induces a map  $\bar{p}$  of  $\mathbb{C}^2/G$  to  $V$ . The map  $\bar{p}$  is easily seen to be injective, and thus is an isomorphism, since  $\mathbb{C}^2/G$  and  $V$  are normal.

The proof for the other finite subgroups  $G$  of  $S^3$  is similar, and may be found in [Du Val 3]: The elements of  $G$  are listed, the subring  $R$  of  $\mathbb{C}[u, v]$  of invariant polynomials is found to be generated by three homogeneous polynomials  $p_1, p_2, p_3$  of various degrees, and they satisfy exactly one weighted homogeneous relation  $f(p_1, p_2, p_3) = 0$ . It follows that  $\mathbb{C}^2/G$  is isomorphic to the zero locus of  $f$ . Special cases of this proof go back to [Klein]. It is also possible to give a simpler uniform proof using vertices, edges, and faces when  $G$  is the commutator subgroup  $[H, H]$  of another finite subgroup  $H$  of  $S^3$  [Milnor 2, §4].

[Du Val 3, §30] gives a geometric description of the links of these singularities as regular solids with opposite faces identified. (The *link* of a germ  $V \subset \mathbb{C}^n$  at  $v$  is  $V$  intersected with a suitably small sphere about  $v$ .)

The finite subgroups of  $GL(2, \mathbb{C})$  are listed in [Du Val 3, §21] and the corresponding quotient singularities are studied in [Brieskorn 2, p. 348]. The ring of invariant polynomials has been computed for the cyclic and dihedral subgroups [Riemenschneider 1,2]. Generalizations of quotient singularities and their relation to weighted homogeneous polynomials may be found in [Milnor 2; Dolgachev].

**Characterization A5'.** The analytic space  $f^{-1}(0)$  is isomorphic to  $\mathbb{C}^2/G$ , where  $G$  is a finite subgroup of  $SU(2)$ .

Proposition 5.2 shows that characterizations A5' and A1 are equivalent. Clearly Characterization A5' implies A5; since A5 implies A2, they are all equivalent.

**COROLLARY 5.3.** *Let  $G$  be a small finite subgroup of  $GL(2, \mathbb{C})$ . Then  $G \subset SL(2, \mathbb{C})$  if and only if  $\mathbb{C}^2/G$  embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

*Fact 1.* Let  $G$  be a small finite subgroup of  $GL(2, \mathbb{C})$ . Then  $G \subset SL(2, \mathbb{C})$  if and only if the singularity of  $\mathbb{C}^2/G$  is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

*Fact 2.* Let  $V$  be the germ at  $\mathbf{v}$  of a two-dimensional rational singularity. Then  $V$  is Gorenstein if and only if  $V$  embeds in codimension 1.

*Proof.* Any singularity embedded in codimension one is Gorenstein. Conversely, suppose  $V$  is Gorenstein. Let  $\pi: M \rightarrow V$  be the minimal resolution of  $V$ , and let  $E_1 \cup \dots \cup E_s = \pi^{-1}(\mathbf{v})$  be its exceptional set as in Section 3. Since  $V$  is Gorenstein, there is a divisor  $K$  on  $M$  (the *canonical class*) satisfying the adjunction formula. Furthermore  $K \cdot E_i \geq 0$  for all  $i$  since the resolution is minimal, so  $K \leq 0$  [Artin, bottom of p. 130]. If  $K < 0$ , then  $-K > 0$ , so arithmetic genus  $p$  of  $-K$  satisfies  $p(-K) \leq 0$  [Artin, Proposition 1]. On the other hand,  $p(-K) = 1 - \chi(-K) = 1$  by the Riemann-Roch Theorem, a contradiction. Hence  $K = 0$ . Thus  $K \cdot E_i = 0$  for all  $i$ , so  $V$  is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

## 6. THE LOCAL FUNDAMENTAL GROUP

Let  $V$  be the germ of a normal two-dimensional complex analytic space with an isolated singularity at  $\mathbf{v}$ . Without loss of generality, we may assume that  $V$  is a *good neighborhood* of  $\mathbf{v}$ , that is, that there is a neighborhood basis  $V_i$  of  $\mathbf{v}$  in  $V$  such that each  $V_i - \mathbf{v}$  is a deformation retract of  $V - \mathbf{v}$  [Prill]. The *local fundamental group* of  $V$  at  $\mathbf{v}$  is then defined as  $\pi_1(V - \mathbf{v})$ . This group is trivial if and only if  $V$  is nonsingular at  $\mathbf{v}$  [Mumford].