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Characterization A2. The singularity of  $f^{-1}(0)$  is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

# 3. Exceptional sets

Let V be as above, and let  $\pi: M \to V$  be a resolution of V. The exceptional set  $E = \pi^{-1}$  (v) is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves  $E_1, ..., E_s$ . It is possible to arrange that the  $E_i$  are non-singular, the intersection of  $E_i$  and  $E_j$  is transverse for  $i \neq j$ , and no three  $E_i$  meet at a point. Such a resolution is called good. If, in addition, the intersection of  $E_i$  and  $E_j$  is empty or one point, the resolution is very good; this is possible to arrange as well.

Suppose that the resolution is good. Let  $E_i \cdot E_j$  equal the number of points of intersection of  $E_i$  and  $E_j$  if  $i \neq j$  (always a non-negative integer), or the first Chern class of the normal bundle to  $E_i$  evaluated on the orientation class of  $E_i$  if i = j (the self-intersection of  $E_i$ ). The matrix  $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution*. It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves  $E = E_1 \cup ... \cup E_s$  in a twodimensional manifold M with negative definite intersection matrix  $\{E_i \cdot E_j\}$ , a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map  $M \to M/E$  is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of  $f^{-1}(0)$  is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

- (i) Let M→V be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor Z = Σn<sub>i</sub>E<sub>i</sub> on M with n<sub>i</sub> ≥ 0 called the *fundamental cycle*, and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic χ(Z) of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.
- (ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].

(iii) A rational singularity V embeds in codimension one if and only if it is a double point, which is true if and only if  $Z^2 = -2$  [Artin, Corollary 6].

 $(A2) \Rightarrow (A3)$ : We only need show  $E_i^2 = -2$  for all *i*. Certainly  $E_i^2 \leq -2$ , since if  $E_i^2 = -1$  the resolution could be contracted by Castelnuovo's criterion, and  $E_i^2 \ge 0$  would contradict the fact that the matrix  $\{E_i \cdot E_j\}$  is negative definite. Let *K* be the canonical class of *M*. (This exists since *V* is Gorenstein; see for instance [Durfee 2].) The adjunction formula  $-E_i \cdot K = E_i^2 + 2$  then shows that  $E_i \cdot K \ge 0$  for each *i*. The Riemann-Roch Theorem  $\chi(Z) = -\frac{1}{2}(Z^2 + Z \cdot K)$  implies that  $Z \cdot K = 0$ . Thus  $0 = Z \cdot K \ge (E_1 + ... + E_s) \cdot K \ge E_i \cdot K \ge 0$ . Hence  $E_i \cdot K = 0$  for all *i*, so again by the adjunction formula,  $E_i^2 = -2$ .

 $(A3) \Rightarrow (A2)$ : The adjunction formula implies that  $E_i \cdot K = 0$  for all *i*; since the matrix  $\{E_i \cdot E_j\}$  is negative definite, K = 0. Thus  $\chi(Z)$  $= \frac{1}{2}Z^2$  by the Riemann-Roch Theorem. Since  $\chi(Z) \leq 1$  and  $Z^2 < 0$ (again since  $\{E_i \cdot E_j\}$  is negative definite),  $\chi(Z)$  must be 1 and  $Z^2$  must be -2. This completes the proof.

Now, exactly what exceptional sets satisfy Characterization A3? First some algebra. It is possible to associate a weighted graph to any symmetric integral bilinear form  $\langle , \rangle$  on a free module with basis  $e_1, ..., e_s$  satisfying  $\langle e_i, e_j \rangle \ge 0$  for  $i \neq j$ : The vertices of the graph are  $v_1, ..., v_s$ , two vertices  $v_i$  and  $v_j$  are joined by  $\langle e_i, e_j \rangle$  edges, and the vertex  $v_i$  is weighted by the integer  $\langle e_i, e_i \rangle$ . Conversely, a weighted graph defines such a bilinear form. Let  $T_{p,q,r}$  be the weighted graph



where p, q, and r are positive integers, and all vertices are weighted by -2.

Lemma 3.1 [Hirzebruch 2, p. 217]. The only connected graphs weighted by -2 and whose associated bilinear form is negative definite are of type  $T_{p,q,r}$ , where p, q, and r are positive integers satisfying  $p^{-1} + q^{-1} + r^{-1} > 1$ .

*Proof.* (a) If the bilinear form associated to a graph is negative definite, so is the bilinear form associated to any subgraph.

(b) The graph ( $s \ge 2$ )



where all vertices  $e_1, ..., e_s$  are weighted by -2, is not negative definite, since  $(e_1 + ... + e_s)^2 = 0$ .

(c) The graph



where all vertices are weighted by -2, is not negative definite, since  $(2e_1 + ... + 2e_s + f_1 + ... + f_4)^2 = 0$ .

Thus the graph must be of the form  $T_{p,q,r}$ . An elementary argument shows that the bilinear form of  $T_{p,q,r}$  is isomorphic over the rationals to the direct sum of a negative definite form and the one-dimensional form  $\langle 1 - p^{-1} - q^{-1} - r^{-1} \rangle$ . Hence  $T_{p,q,r}$  is negative definite if and only if  $p^{-1} + q^{-1} + r^{-1} > 1$ . This proves the lemma.

The only triples of positive integers (p, q, r) satisfying  $p^{-1} + q^{-1} + r^{-1} > 1$  are of course just (1, 1, r) for  $r \ge 1$ , (2, 2, r) for  $r \ge 2$ , (2, 3, 3), (2, 3, 4), and (2, 3, 5).

The *dual graph of a resolution of a singularity* is defined to be the weighted graph associated to the intersection matrix of the resolution. Applying the above facts, we see that Characterization A3 is equivalent to:

Characterization A3'. The minimal resolution of  $f^{-1}(0)$  is listed in column (3) of Table 1.

Next we show that Characterization A1 and A3 are equivalent. Characterization A1 implies Characterization A3 since the singularities of the — 137 —

functions f listed in column 1 of Table 1 have minimal resolutions as in column 3. (I believe that this first appeared in [Hirzebruch 1].) The converse follows since the singularities listed are taut [Brieskorn 2; Tjurina 3; Laufer 4]. (Two resolutions  $\pi: M \to V$  and  $\pi': M' \to V'$  are topologically equivalent if their exceptional sets are homeomorphic by a homeomorphism preserving the self-intersection numbers. A singularity V is taut if any other singularity with a good resolution topologically equivalent to a good resolution of V is then isomorphic to V.)

The classification of rational double points has been generalized in several ways: to rational triple points [Artin, p. 135], to elliptic singularities [Wagreich 1], and to minimally elliptic singularities [Laufer 5]. The Dynkin diagrams  $B_n$ ,  $C_n$ ,  $F_4$  and  $G_2$  occur when resolving singularities over non-algebraically closed fields [Lipman 1]. There is also a relation with simple complex Lie groups [Brieskorn 3].

## 4. Absolutely isolated double points

There are at least three methods of resolving the singularity of the germ of a normal two-dimensional complex space V. The first method is one of local uniformization; this is originally due to Jung, and is described in detail in [Laufer 1]. The second method, due to Zariski, is to alternately blow up points and normalize. The third method (which generalizes to higher dimensions), is to blow up points and non-singular curves.

The singularity of V is *absolutely isolated* if it may be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. For example, the singularity of the zero locus of  $f(x, y, z) = x^k + y^k + z^k$ is absolutely isolated, since it may be resolved by blowing up the origin once.

The singularity of V is a *double point* if its local ring is of multiplicity two. If V is  $f^{-1}(0)$ , this is equivalent to the lowest non-zero homogeneous term in the power series expansion of f being quadratic.

Characterization A4. The singularity of  $f^{-1}(0)$  is an absolutely isolated double point.

The equivalence of Characterizations A1 and A4 was proved directly in [Kirby]. Later, it was shown [Tjurina 2; Lipman 1] that all rational singularities are absolutely isolated (thus showing Characterization A2 implies A4), and in [Brieskorn 1, Satz 1] that A4 implies A3.