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AND SIMPLE CRITICAL POINTS

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Characterization A2. The singularity of $f^{-1}(0)$ is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

3. EXCEPTIONAL SETS

Let V be as above, and let $\pi: M \to V$ be a resolution of V. The exceptional set $E = \pi^{-1}$ (v) is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves $E_1, ..., E_s$. It is possible to arrange that the E_i are non-singular, the intersection of E_i and E_j is transverse for $i \neq j$, and no three E_i meet at a point. Such a resolution is called good. If, in addition, the intersection of E_i and E_j is empty or one point, the resolution is very good; this is possible to arrange as well.

Suppose that the resolution is good. Let $E_i \cdot E_j$ equal the number of points of intersection of E_i and E_j if $i \neq j$ (always a non-negative integer), or the first Chern class of the normal bundle to E_i evaluated on the orientation class of E_i if i=j (the self-intersection of E_i). The matrix $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution*. It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves $E = E_1 \cup ... \cup E_s$ in a two-dimensional manifold M with negative definite intersection matrix $\{E_i \cdot E_j\}$, a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map $M \to M/E$ is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of $f^{-1}(0)$ is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

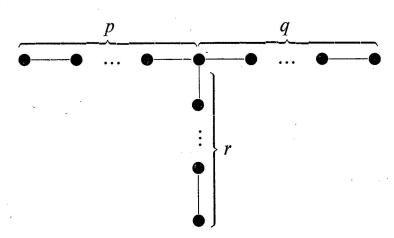
- (i) Let $M \to V$ be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor $Z = \sum n_i E_i$ on M with $n_i \ge 0$ called the *fundamental cycle*, and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic $\chi(Z)$ of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.
- (ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].

(iii) A rational singularity V embeds in codimension one if and only if it is a double point, which is true if and only if $Z^2 = -2$ [Artin, Corollary 6].

 $(A2)\Rightarrow (A3)$: We only need show $E_i^2=-2$ for all i. Certainly $E_i^2\leqslant -2$, since if $E_i^2=-1$ the resolution could be contracted by Castelnuovo's criterion, and $E_i^2\geqslant 0$ would contradict the fact that the matrix $\{E_i\cdot E_j\}$ is negative definite. Let K be the canonical class of M. (This exists since V is Gorenstein; see for instance [Durfee 2].) The adjunction formula $-E_i\cdot K=E_i^2+2$ then shows that $E_i\cdot K\geqslant 0$ for each i. The Riemann-Roch Theorem $\chi(Z)=-\frac{1}{2}(Z^2+Z\cdot K)$ implies that $Z\cdot K=0$. Thus $0=Z\cdot K\geqslant (E_1+...+E_s)\cdot K\geqslant E_i\cdot K\geqslant 0$. Hence $E_i\cdot K=0$ for all i, so again by the adjunction formula, $E_i^2=-2$.

 $(A3) \Rightarrow (A2)$: The adjunction formula implies that $E_i \cdot K = 0$ for all i; since the matrix $\{E_i \cdot E_j\}$ is negative definite, K = 0. Thus $\chi(Z) = \frac{1}{2}Z^2$ by the Riemann-Roch Theorem. Since $\chi(Z) \leq 1$ and $Z^2 < 0$ (again since $\{E_i \cdot E_j\}$ is negative definite), $\chi(Z)$ must be 1 and Z^2 must be -2. This completes the proof.

Now, exactly what exceptional sets satisfy Characterization A3? First some algebra. It is possible to associate a weighted graph to any symmetric integral bilinear form \langle , \rangle on a free module with basis $e_1, ..., e_s$ satisfying $\langle e_i, e_j \rangle \geqslant 0$ for $i \neq j$: The vertices of the graph are $v_1, ..., v_s$, two vertices v_i and v_j are joined by $\langle e_i, e_j \rangle$ edges, and the vertex v_i is weighted by the integer $\langle e_i, e_i \rangle$. Conversely, a weighted graph defines such a bilinear form. Let $T_{p,q,r}$ be the weighted graph

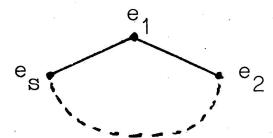


where p, q, and r are positive integers, and all vertices are weighted by -2.

Lemma 3.1 [Hirzebruch 2, p. 217]. The only connected graphs weighted by -2 and whose associated bilinear form is negative definite are of type $T_{p,q,r}$, where p, q, and r are positive integers satisfying $p^{-1} + q^{-1} + r^{-1} > 1$.

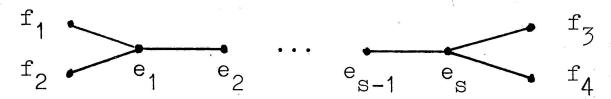
Proof. (a) If the bilinear form associated to a graph is negative definite, so is the bilinear form associated to any subgraph.

(b) The graph $(s \ge 2)$



where all vertices $e_1, ..., e_s$ are weighted by -2, is not negative definite, since $(e_1 + ... + e_s)^2 = 0$.

(c) The graph



where all vertices are weighted by -2, is not negative definite, since $(2e_1 + ... + 2e_s + f_1 + ... + f_4)^2 = 0$.

Thus the graph must be of the form $T_{p,q,r}$. An elementary argument shows that the bilinear form of $T_{p,q,r}$ is isomorphic over the rationals to the direct sum of a negative definite form and the one-dimensional form $(1-p^{-1}-q^{-1}-r^{-1})$. Hence $T_{p,q,r}$ is negative definite if and only if $p^{-1}+q^{-1}+r^{-1}>1$. This proves the lemma.

The only triples of positive integers (p, q, r) satisfying $p^{-1} + q^{-1} + r^{-1} > 1$ are of course just (1, 1, r) for $r \ge 1$, (2, 2, r) for $r \ge 2$, (2, 3, 3), (2, 3, 4), and (2, 3, 5).

The dual graph of a resolution of a singularity is defined to be the weighted graph associated to the intersection matrix of the resolution. Applying the above facts, we see that Characterization A3 is equivalent to:

Characterization A3'. The minimal resolution of $f^{-1}(0)$ is listed in column (3) of Table 1.

Next we show that Characterization A1 and A3 are equivalent. Characterization A1 implies Characterization A3 since the singularities of the

functions f listed in column 1 of Table 1 have minimal resolutions as in column 3. (I believe that this first appeared in [Hirzebruch 1].) The converse follows since the singularities listed are taut [Brieskorn 2; Tjurina 3; Laufer 4]. (Two resolutions $\pi: M \to V$ and $\pi': M' \to V'$ are topologically equivalent if their exceptional sets are homeomorphic by a homeomorphism preserving the self-intersection numbers. A singularity V is taut if any other singularity with a good resolution topologically equivalent to a good resolution of V is then isomorphic to V.)

The classification of rational double points has been generalized in several ways: to rational triple points [Artin, p. 135], to elliptic singularities [Wagreich 1], and to minimally elliptic singularities [Laufer 5]. The Dynkin diagrams B_n , C_n , F_4 and G_2 occur when resolving singularities over non-algebraically closed fields [Lipman 1]. There is also a relation with simple complex Lie groups [Brieskorn 3].

4. Absolutely isolated double points

There are at least three methods of resolving the singularity of the germ of a normal two-dimensional complex space V. The first method is one of local uniformization; this is originally due to Jung, and is described in detail in [Laufer 1]. The second method, due to Zariski, is to alternately blow up points and normalize. The third method (which generalizes to higher dimensions), is to blow up points and non-singular curves.

The singularity of V is absolutely isolated if it may be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. For example, the singularity of the zero locus of $f(x, y, z) = x^k + y^k + z^k$ is absolutely isolated, since it may be resolved by blowing up the origin once.

The singularity of V is a *double point* if its local ring is of multiplicity two. If V is f^{-1} (0), this is equivalent to the lowest non-zero homogeneous term in the power series expansion of f being quadratic.

Characterization A4. The singularity of $f^{-1}(0)$ is an absolutely isolated double point.

The equivalence of Characterizations A1 and A4 was proved directly in [Kirby]. Later, it was shown [Tjurina 2; Lipman 1] that all rational singularities are absolutely isolated (thus showing Characterization A2 implies A4), and in [Brieskorn 1, Satz 1] that A4 implies A3.