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germs  $V$  and  $W$  embedded in  $\mathbf{C}^n$  at the origin are *isomorphic* if there is a germ of an analytic automorphism of  $\mathbf{C}^n$  fixing the origin and taking  $V$  to  $W$ .

*Characterization A1.* The analytic set  $f^{-1}(0)$  is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

## 2. RATIONAL SINGULARITIES

A *resolution* of a germ of a normal surface singularity  $V$  as above is a complex analytic manifold  $M$  and an analytic map  $\pi: M \rightarrow V$  that is surjective and proper (compact fibers) such that its restriction to  $M - \pi^{-1}(\mathbf{v})$  is an analytic isomorphism, and  $M - \pi^{-1}(\mathbf{v})$  is dense in  $M$ . Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution*  $\pi: M \rightarrow V$  that has the following universal mapping property: Given any other resolution  $\pi': M' \rightarrow V$ , there is a unique map  $\rho: M' \rightarrow M$  with  $\pi' = \pi \circ \rho$ .

The *geometric genus*  $p$  of  $V$  is the dimension of the complex vector space  $H^1(M, \mathcal{O}_M)$ , where  $M$  is any resolution of  $V$ , and  $\mathcal{O}_M$  is the sheaf of holomorphic functions on  $M$  [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. ( $V$  is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf  $R^1 \pi_* \mathcal{O}_M$  on  $V$ . The idea behind this definition is that  $M$  is a collection of "thickened" curves, and that the genus of a curve  $X$  is the dimension of  $H^1(X, \mathcal{O}_X)$ . For example,  $H^1(M, \mathcal{O}_M) = 0$  if  $M$  is the total space of a line bundle over a curve of genus zero. On the other hand,  $\dim H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$  if  $M$  is a line bundle of Chern class  $-k$  over a curve of genus  $(k-1)(k-2)/2$  (the minimal resolution of  $f(x, y, z) = x^k + y^k + z^k$ ). In terms of  $V$  alone,  $p$  is the dimension of the space of holomorphic two-forms on  $V - \mathbf{v}$  divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for  $p$  in terms of topological invariants of the resolution  $M$  and the nearby fiber  $F$  (see §11) is given in [Laufer 6].

The analytic set  $V$  has a *rational singularity* if  $p = 0$ . A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].

*Characterization A2.* The singularity of  $f^{-1}(0)$  is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

### 3. EXCEPTIONAL SETS

Let  $V$  be as above, and let  $\pi: M \rightarrow V$  be a resolution of  $V$ . The *exceptional set*  $E = \pi^{-1}(\mathbf{v})$  is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves  $E_1, \dots, E_s$ . It is possible to arrange that the  $E_i$  are non-singular, the intersection of  $E_i$  and  $E_j$  is transverse for  $i \neq j$ , and no three  $E_i$  meet at a point. Such a resolution is called *good*. If, in addition, the intersection of  $E_i$  and  $E_j$  is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let  $E_i \cdot E_j$  equal the number of points of intersection of  $E_i$  and  $E_j$  if  $i \neq j$  (always a non-negative integer), or the first Chern class of the normal bundle to  $E_i$  evaluated on the orientation class of  $E_i$  if  $i = j$  (the self-intersection of  $E_i$ ). The matrix  $\{E_i \cdot E_j\}$  is called the *intersection matrix of the resolution*. It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves  $E = E_1 \cup \dots \cup E_s$  in a two-dimensional manifold  $M$  with negative definite intersection matrix  $\{E_i \cdot E_j\}$ , a theorem of Grauert says that the quotient space  $M/E$  has a normal complex structure and that the projection map  $M \rightarrow M/E$  is analytic [Laufer 1, p. 60].

*Characterization A3.* The minimal resolution of  $f^{-1}(0)$  is very good, and its exceptional set consists of curves of genus 0 and self-intersection  $-2$ .

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

- (i) Let  $M \rightarrow V$  be a resolution of a normal singularity  $V$  as above. There is a certain unique non-zero divisor  $Z = \sum n_i E_i$  on  $M$  with  $n_i \geq 0$  called the *fundamental cycle*, and it is shown that the singularity of  $V$  is rational if and only if the analytic Euler characteristic  $\chi(Z)$  of  $Z$  is 1 (that is, the arithmetic genus of  $Z$  is 0) [Artin, Theorem 3]. It is easy to see that the support of  $Z$  is the whole exceptional set of  $E$ .
- (ii) Any resolution of a rational singularity  $V$  is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].