fifteen characterizations, since Characterization A1 coincides with Characterization B2.

Most of the characterizations of Part B are shown to be equivalent to Characterization B1. Other links between the two sets of characterizations are provided by Theorem 12.2, which shows that Characterizations A2 and B5 are equivalent, and a recent result (Theorem 11.1) partially connecting Characterizations A2 and B3. Part B also contains a summary of pertinent work of Mather and Arnold.

There are two appendices. The first gives nine characterizations of simple elliptic singularities and almost-simple critical points. They are the next most reasonable class of singularities after rational double points, and can be characterized as being "infinite but not too infinite". All remaining singularities are "very infinite" in various senses. The second appendix contains Looijenga's proof that the monodromy group of the minimal hyperbolic germs has exponential growth.

This paper is an expanded version of a series of lectures given at the University of Maryland in the spring of 1976, and I thank the department of mathematics for its hospitality. The lectures were inspired by an unpublished talk given by E. Brieskorn at the American Mathematical Society Summer Institute in Algebraic Geometry in Arcata (1974). I also thank E. Looijenga and J. Wahl for helpful comments.

A. Seven characterizations of rational double points

THEOREM A. *Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f. Then characterizations A1 through A7 (which are listed below) are equivalent.*

1. Complex analytic spaces

Let V be the germ at $\mathbf{v}$ of a normal two-dimensional complex analytic space with a singularity at $\mathbf{v}$. (The definitions of these terms can be found in [Laufer 1].) For example, V could be $f^{-1}(0)$, where f is as in the hypotheses of Theorem A. Conversely, if V is embedded in $\mathbf{C}^3$ with $\mathbf{v}$ the origin, there is a germ f as above such that V is isomorphic to $f^{-1}(0)$ [Gunning and Rossi, p. 113]. The singularity is isolated since V is normal. Two such germs V and W embedded in $\mathbf{C}^n$ at the origin are *isomorphic* if there is a germ of an analytic automorphism of $\mathbf{C}^n$ fixing the origin and taking V to W.

Characterization A1. The analytic set $f^{-1}(0)$ is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

2. Rational singularities

A *resolution* of a germ of a normal surface singularity V as above is a complex analytic manifold M and an analytic map $\pi: M \to V$ that is surjective and proper (compact fibers) such that its restriction to $M - \pi^{-1}(\mathbf{v})$ is an analytic isomorphism, and $M - \pi^{-1}(\mathbf{v})$ is dense in M. Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution* $\pi: M \to V$ that has the following universal mapping property: Given any other resolution $\pi': M' \to V$, there is a unique map $\rho: M' \to M$ with $\pi' = \pi \circ \rho$.

The *geometric genus p* of V is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$, where M is any resolution of V, and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. (V is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf $R^1 \pi_* \mathcal{O}_M$ on V. The idea behind this definition is that M is a collection of "thickened" curves, and that the genus of a curve X is the dimension of $H^1(X, \mathcal{O}_X)$. For example, $H^1(M, \mathcal{O}_M) = 0$ if M is the total space of a line bundle over a curve of genus zero. On the other hand, dim $H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$ if M is a line bundle of Chern class $-k$ over a curve of genus $(k-1)(k-2)/2$ (the minimal resolution of $f(x, y, z) = x^k + y^k + z^k$). In terms of V alone, p is the dimension of the space of holomorphic two-forms on $V - \mathbf{v}$ divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for p in terms of topological invariants of the resolution M and the nearby fiber F (see §11) is given in [Laufer 6].

The analytic set V has a *rational* singularity if $p = 0$. A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].

Characterization A2. The singularity of $f^{-1}(0)$ is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

3. Exceptional sets

Let V be as above, and let $\pi: M \to V$ be a resolution of V. The *exceptional set* $E = \pi^{-1}(\mathbf{v})$ is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves $E_1, \ldots, E_s$. It is possible to arrange that the E_i are non-singular, the intersection of E_i and E_j is transverse for $i \neq j$, and no three E_i meet at a point. Such a resolution is called *good.* If, in addition, the intersection of E_i and E_j is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let $E_i \cdot E_j$ equal the number of points of intersection of E_i and E_j if $i \neq j$ (always a non-negative integer), or the first Chern class of the normal bundle to E_i evaluated on the orientation class of E_i if $i = j$ (the self-intersection of E_i). The matrix $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution.* It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves $E = E_1 \cup \ldots \cup E_s$ in a two-dimensional manifold M with negative definite intersection matrix $\{E_i \cdot E_j\}$, a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map $M \to M/E$ is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of $f^{-1}(0)$ is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

(i) Let $M \to V$ be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor $Z = \Sigma n_i E_i$ on M with $n_i \geqslant 0$ called the *fundamental cycle,* and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic $\chi(Z)$ of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.

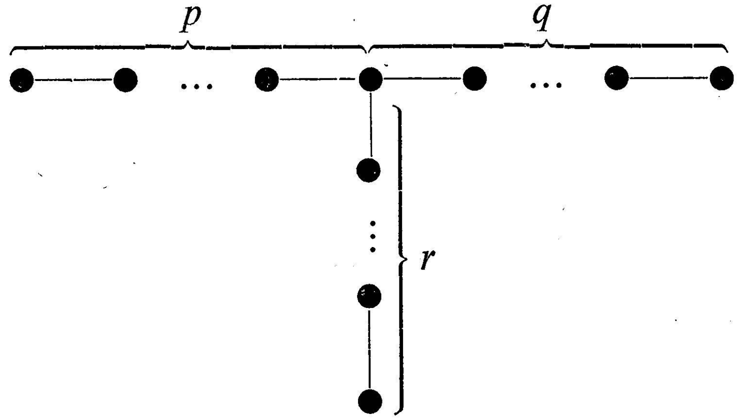
(ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].

(iii) A rational singularity V embeds in codimension one if and only if it is a double point, which is true if and only if $Z^2 = -2$ [Artin, Corollary 6].

(A2) $\Rightarrow$ *(A3)*: We only need show $E_i^2 = -2$ for all i. Certainly $E_i^2 \leqslant -2$, since if $E_i^2 = -1$ the resolution could be contracted by Castelnuovo's criterion, and $E_i^2 \geqslant 0$ would contradict the fact that the matrix $\{E_i \cdot E_j\}$ is negative definite. Let K be the canonical class of M. (This exists since V is Gorenstein; see for instance [Durfee 2].) The adjunction formula $-E_i \cdot K = E_i^2 + 2$ then shows that $E_i \cdot K \geqslant 0$ for each i. The Riemann-Roch Theorem $\chi(Z) = -\frac{1}{2}(Z^2 + Z \cdot K)$ implies that $Z \cdot K = 0$. Thus $0 = Z \cdot K \geqslant (E_1 + \ldots + E_s) \cdot K \geqslant E_i \cdot K \geqslant 0$. Hence $E_i \cdot K = 0$ for all i, so again by the adjunction formula, $E_i^2 = -2$.

(A3) $\Rightarrow$ *(A2)*: The adjunction formula implies that $E_i \cdot K = 0$ for all i; since the matrix $\{E_i \cdot E_j\}$ is negative definite, $K = 0$. Thus $\chi(Z) = \frac{1}{2} Z^2$ by the Riemann-Roch Theorem. Since $\chi(Z) \leqslant 1$ and $Z^2 < 0$ (again since $\{E_i \cdot E_j\}$ is negative definite), $\chi(Z)$ must be 1 and Z^2 must be -2. This completes the proof.

Now, exactly what exceptional sets satisfy Characterization A3? First some algebra. It is possible to associate a weighted graph to any symmetric integral bilinear form $\langle\, , \rangle$ on a free module with basis $e_1, \ldots, e_s$ satisfying $\langle e_i, e_j \rangle \geqslant 0$ for $i \neq j$: The vertices of the graph are $v_1, \ldots, v_s$, two vertices v_i and v_j are joined by $\langle e_i, e_j \rangle$ edges, and the vertex v_i is weighted by the integer $\langle e_i, e_i \rangle$. Conversely, a weighted graph defines such a bilinear form. Let $T_{p,q,r}$ be the weighted graph

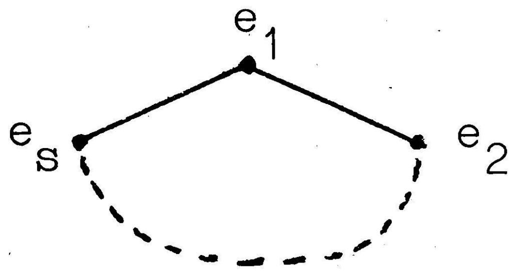


where p, q, and r are positive integers, and all vertices are weighted by -2.

Lemma 3.1 [Hirzebruch 2, p. 217]. The only connected graphs weighted by -2 and whose associated bilinear form is negative definite are of type $T_{p,q,r}$, where p, q, and r are positive integers satisfying $p^{-1} + q^{-1} + r^{-1} > 1$.

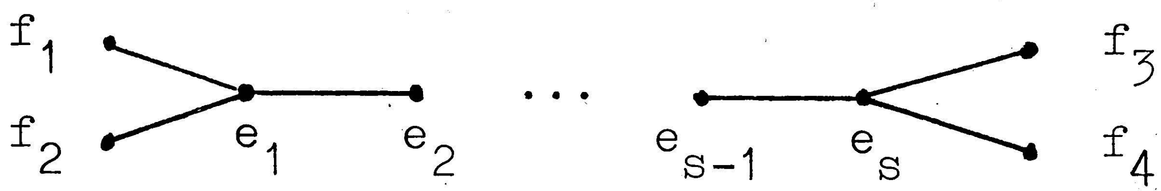
Proof. (a) If the bilinear form associated to a graph is negative definite, so is the bilinear form associated to any subgraph.

(b) The graph ($s \geqslant 2$)



where all vertices $e_1, \ldots, e_s$ are weighted by -2, is not negative definite, since $(e_1 + \ldots + e_s)^2 = 0$.

(c) The graph



where all vertices are weighted by -2, is not negative definite, since $(2e_1 + \ldots + 2e_s + f_1 + \ldots + f_4)^2 = 0$.

Thus the graph must be of the form $T_{p,q,r}$. An elementary argument shows that the bilinear form of $T_{p,q,r}$ is isomorphic over the rationals to the direct sum of a negative definite form and the one-dimensional form $\langle 1 - p^{-1} - q^{-1} - r^{-1} \rangle$. Hence $T_{p,q,r}$ is negative definite if and only if $p^{-1} + q^{-1} + r^{-1} > 1$. This proves the lemma.

The only triples of positive integers (p, q, r) satisfying $p^{-1} + q^{-1} + r^{-1} > 1$ are of course just $(1, 1, r)$ for $r \geqslant 1$, $(2, 2, r)$ for $r \geqslant 2$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$.

The *dual graph of a resolution of a singularity* is defined to be the weighted graph associated to the intersection matrix of the resolution. Applying the above facts, we see that Characterization A3 is equivalent to:

Characterization A3'. The minimal resolution of $f^{-1}(0)$ is listed in column (3) of Table 1.

Next we show that Characterization A1 and A3 are equivalent. Characterization A1 implies Characterization A3 since the singularities of the functions f listed in column 1 of Table 1 have minimal resolutions as in column 3. (I believe that this first appeared in [Hirzebruch 1].) The converse follows since the singularities listed are taut [Brieskorn 2; Tjurina 3; Laufer 4]. (Two resolutions $\pi: M \to V$ and $\pi': M' \to V'$ are *topologically equivalent* if their exceptional sets are homeomorphic by a homeomorphism preserving the self-intersection numbers. A singularity V is *taut* if any other singularity with a good resolution topologically equivalent to a good resolution of V is then isomorphic to V.)

The classification of rational double points has been generalized in several ways: to rational triple points [Artin, p. 135], to elliptic singularities [Wagreich 1], and to minimally elliptic singularities [Laufer 5]. The Dynkin diagrams B_n, C_n, F_4 and G_2 occur when resolving singularities over non-algebraically closed fields [Lipman 1]. There is also a relation with simple complex Lie groups [Brieskorn 3].

4. Absolutely isolated double points

There are at least three methods of resolving the singularity of the germ of a normal two-dimensional complex space V. The first method is one of local uniformization; this is originally due to Jung, and is described in detail in [Laufer 1]. The second method, due to Zariski, is to alternately blow up points and normalize. The third method (which generalizes to higher dimensions), is to blow up points and non-singular curves.

The singularity of V is *absolutely isolated* if it may be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. For example, the singularity of the zero locus of $f(x, y, z) = x^k + y^k + z^k$ is absolutely isolated, since it may be resolved by blowing up the origin once.

The singularity of V is a *double point* if its local ring is of multiplicity two. If V is $f^{-1}(0)$, this is equivalent to the lowest non-zero homogeneous term in the power series expansion of f being quadratic.

Characterization A4. The singularity of $f^{-1}(0)$ is an absolutely isolated double point.

The equivalence of Characterizations A1 and A4 was proved directly in [Kirby]. Later, it was shown [Tjurina 2; Lipman 1] that all rational singularities are absolutely isolated (thus showing Characterization A2 implies A4), and in [Brieskorn 1, Satz 1] that A4 implies A3.

5. Quotient Singularities

Let U be a neighborhood of the origin $\mathbf{0}$ in $\mathbf{C}^2$ and let H be a finite group of analytic automorphisms of U fixing $\mathbf{0}$. The quotient space U/H has the structure of a normal two-dimensional complex analytic space with an isolated singularity, and the projection map $U \to U/H$ is analytic [Cartan]. An analytic space V is called a *quotient singularity* if there is a U and H as above such that V is isomorphic to U/H.

An important example of a quotient singularity is $\mathbf{C}^2/G$, where G is some finite subgroup of $GL(2, \mathbf{C})$. The space $\mathbf{C}^2/G$ is not just analytic, but algebraic. For any finite subgroup G of $GL(2, \mathbf{C})$, the ring of functions on the algebraic variety $\mathbf{C}^2/G$ is isomorphic to the subring of invariant polynomials in $GL(2, \mathbf{C})$. Hence to find $\mathbf{C}^2/G$ it suffices to find this subring of invariant polynomials. Note that a finite subgroup G of $GL(2, \mathbf{C})$ or $SL(2, \mathbf{C})$ is conjugate to a finite subgroup of $U(2)$ or $SU(2)$ respectively, since it is possible to choose an invariant Hermitian metric on $\mathbf{C}^2$. A subgroup $G \subset GL(2, \mathbf{C})$ is *small* if no $g \in G$ has 1 as an eigenvalue of multiplicity one. [Prill, p. 380].

Proposition 5.1. *Let V be the germ of a normal two-dimensional complex analytic space. The following statements are equivalent.*

(a) *V is a quotient singularity.*

(b) *V is isomorphic to $\mathbf{C}^2/G$, for some finite subgroup G of* $GL(2, \mathbf{C})$.

(c) *V is isomorphic to $\mathbf{C}^2/G$, for some small finite subgroup of* $GL(2, \mathbf{C})$.

Condition (a) implies condition (b) by the usual linearization argument [Brieskorn 2, Lemma 2.2]. It is shown in [Prill, p. 380] that condition (b) implies condition (c). Obviously (c) implies (a). The following theorem is also proved in [Prill]: Let G and G' be small finite subgroups of $GL(2, \mathbf{C})$. Then the analytic spaces $\mathbf{C}^2/G$ and $\mathbf{C}^2/G'$ are isomorphic if and only if G and G' are conjugate.

Characterization A5. The analytic space $f^{-1}(0)$ is a quotient singularity.

Since quotient singularities are rational [Brieskorn 2, p. 340], Characterization A5 implies Characterization A2. The converse will follow in round-about fashion.

Consider $SU(2)$, which is of course isomorphic to the group S^3 of unit quaternions. The finite subgroups of S^3 are the cyclic group and the inverse images of the finite subgroups of the rotation group $SO(3)$ under the double cover $S^3 \to SO(3)$; these groups are listed in column 5 of Table 1.

PROPOSITION 5.2. *Let G be a non-trivial finite subgroup of $SU(2)$ as listed in column 5 of Table* 1. *Then* $\mathbf{C}^2/G$ *is isomorphic to* $f^{-1}(0)$, *where f is the corresponding polynomial in column 1.*

In particular, for each polynomial f in column 1 of Table 1 the analytic space $f^{-1}(0)$ is isomorphic to a quotient singularity. This proposition is proved by classical invariant theory. For the cyclic group it is easy: Let $G \subset SU(2)$ be the cyclic group of order k, generated by the transformation $(u, v) \to (\eta u, \eta^{-1} v)$ where η is a primitive k-th root of unity. Then we claim that $\mathbf{C}^2/G$ is isomorphic to

$$V = \{ (x, y, z) \in \mathbf{C}^3 : x^k = yz \} .$$

Let $p_1(u, v) = uv$, $p_2(u, v) = u^k$, $p_3(u, v) = v^k$, and let $p = (p_1, p_2, p_3)$ define a map of $\mathbf{C}^2$ to $\mathbf{C}^3$. The image of p is exactly V. Since $p_i(gu, gv) = p_i(u, v)$ for all g in G, the map p induces a map $\bar{p}$ of $\mathbf{C}^2/G$ to V. The map $\bar{p}$ is easily seen to be injective, and thus is an isomorphism, since $\mathbf{C}^2/G$ and V are normal.

The proof for the other finite subgroups G of S^3 is similar, and may be found in [Du Val 3]: The elements of G are listed, the subring R of $\mathbf{C}[u, v]$ of invariant polynomials is found to be generated by three homogeneous polynomials p_1, p_2, p_3 of various degrees, and they satisfy exactly one weighted homogeneous relation $f(p_1, p_2, p_3) = 0$. It follows that $\mathbf{C}^2/G$ is isomorphic to the zero locus of f. Special cases of this proof go back to [Klein]. It is also possible to give a simpler uniform proof using vertices, edges, and faces when G is the commutator subgroup $[H, H]$ of another finite subgroup H of S^3 [Milnor 2, §4].

[Du Val 3, §30] gives a geometric description of the links of these singularities as regular solids with opposite faces identified. (The *link* of a germ $V \subset \mathbf{C}^n$ at $\mathbf{v}$ is V intersected with a suitably small sphere about $\mathbf{v}$.)

The finite subgroups of $GL(2, \mathbf{C})$ are listed in [Du Val 3, §21] and the corresponding quotient singularities are studied in [Brieskorn 2, p. 348]. The ring of invariant polynomials has been computed for the cyclic and dihedral subgroups [Riemenschneider 1,2]. Generalizations of quotient singularities and their relation to weighted homogeneous polynomials may be found in [Milnor 2; Dolgachev].

Characterization A5′. The analytic space $f^{-1}(0)$ is isomorphic to $\mathbf{C}^2/G$, where G is a finite subgroup of $SU(2)$.

Proposition 5.2 shows that characterizations A5′ and A1 are equivalent. Clearly Characterization A5′ implies A5; since A5 implies A2, they are all equivalent.

Corollary 5.3. *Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if $\mathbf{C}^2/G$ embeds in codimension one.*

This corollary follows from the above case-by-case analysis. J. Wahl points out that it is also possible to prove it directly, using the following two facts:

Fact 1. Let G be a small finite subgroup of $GL(2, \mathbf{C})$. Then $G \subset SL(2, \mathbf{C})$ if and only if the singularity of $\mathbf{C}^2/G$ is Gorenstein.

This is a special case of [Watanabe]. A germ of a normal two-dimensional complex space is *Gorenstein* if there is a nowhere-vanishing holomorphic two-form on its regular points.

Fact 2. Let V be the germ at $\mathbf{v}$ of a two-dimensional rational singularity. Then V is Gorenstein if and only if V embeds in codimension 1.

Proof. Any singularity embedded in codimension one is Gorenstein. Conversely, suppose V is Gorenstein. Let $\pi: M \to V$ be the minimal resolution of V, and let $E_1 \cup \ldots \cup E_s = \pi^{-1}(v)$ be its exceptional set as in Section 3. Since V is Gorenstein, there is a divisor K on M (the *canonical class*) satisfying the adjunction formula. Furthermore $K \cdot E_i \geqslant 0$ for all i since the resolution is minimal, so $K \leqslant 0$ [Artin, bottom of p. 130]. If $K < 0$, then $-K > 0$, so arithmetic genus p of $-K$ satisfies $p(-K) \leqslant 0$ [Artin, Proposition 1]. On the other hand, $p(-K) = 1 - \chi(-K) = 1$ by the Riemann-Roch Theorem, a contradiction. Hence $K = 0$. Thus $K \cdot E_i = 0$ for all i, so V is a double point and embeds in codimension one, as in the proof that Characterization A3 implies Characterization A2.

6. The local fundamental group

Let V be the germ of a normal two-dimensional complex analytic space with an isolated singularity at $\mathbf{v}$. Without loss of generality, we may assume that V is a *good neighborhood* of $\mathbf{v}$, that is, that there is a neighborhood basis V_i of $\mathbf{v}$ in V such that each $V_i - \mathbf{v}$ is a deformation retract of $V - \mathbf{v}$ [Prill]. The *local fundamental group* of V at $\mathbf{v}$ is then defined as $\pi_1(V - \mathbf{v})$. This group is trivial if and only if V is nonsingular at $\mathbf{v}$ [Mumford].

PROPOSITION 5.1 (bis). *The following statement is equivalent to those listed above.*

(d) *The local fundamental group of V is finite.*

It is shown in [Prill, p. 381; Brieskorn 2, p. 344] that conditions (a) and (d) are equivalent.

Characterization A6. The local fundamental group of $f^{-1}(0)$ is finite.

Thus Characterizations A5 and A6 are equivalent.

There is an algorithm for computing the local fundamental group of V from a resolution [Mumford], and singularities V with finite, nilpotent and solvable local fundamental group have been classified [Brieskorn 2; Wagreich 2]. When V is a complete intersection, this classification is particularly simple [Durfee 2, Proposition 3.3].

7. VOLUME

Let $f(x, y, z)$ be the germ at the origin $\mathbf{0}$ of a complex analytic function, and suppose that $f(\mathbf{0}) = 0$ and that the origin is an isolated critical point of f. There is an $\varepsilon > 0$ such that $f^{-1}(\mathbf{0})$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. (See Section 12.) For $t \in \mathbf{C}$, let

$$V_t = f^{-1}(t) \cap D_\varepsilon^6$$

where D_ε^6 is the closed disk of radius ε about $\mathbf{0}$. The function $f(x, y, z)$ takes the constant value t on V_t, so $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \equiv 0$ there. Hence a nowhere-vanishing holomorphic two-form ω_t on V_t may be defined by the equivalent expressions

$$\omega_t = \frac{dy \wedge dz}{\partial f / \partial x} = \frac{dz \wedge dx}{\partial f / \partial y} = \frac{dx \wedge dy}{\partial f / \partial z},$$

Characterization A7. The integral $\int_{V_0} \omega_0 \wedge \bar{\omega}_0$ is finite.

Note that the form $\omega_0 \wedge \bar{\omega}_0$ takes positive real values. The equivalence of Characterizations A2 and A7 is due to Laufer, and follows easily from his expression for the geometric genus in terms of forms [Laufer 2, Corollary 3.6].

A different formulation of this characterization is due to E. Looijenga (unpublished): Let $\Delta(r) = \{t \in \mathbf{C} : t < r\}$, let

$$X(r) = f^{-1}(\Delta(r)) \cap D_\varepsilon^6$$

and let vol $(X(r))$ be its volume in $\mathbf{C}^3$.

Characterization A7'. $\lim_{r \to 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of $\mathbf{C}^3$. Characterizations A7 and A7' are equivalent since

$$\lim_{r \to 0} \frac{1}{r^2} \operatorname{vol}(X(r)) = \lim_{r \to 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega}$$

$$= \lim_{r \to 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \overline{dt},$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2}\right) dt \wedge \overline{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0 .$$

B. Nine characterizations of simple critical points

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \ldots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

Theorem B. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 1$ be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that* $\mathbf{0}$ *is an isolated critical point of f. Then Characterizations B1 through B9 are equivalent.*

8. The classification of right equivalence classes

Let $\mathcal{O}$ be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on $\mathbf{C}^{n+1}$. (In other words, $\mathcal{O}$ is just the ring $\mathbf{C}\{z_0, \ldots, z_n\}$ of convergent power series.) The ring $\mathcal{O}$ is local with maximal ideal

$$\mathfrak{m} = \{ f \in \mathcal{O} : f(\mathbf{0}) = 0 \}.$$