Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS
Autor: Bierstone, Edward / Milman, Pierre
Kapitel: 4. A PROJECTION FORMULA
DOI: https://doi.org/10.5169/seals-50374

Remark 3.2. There are more precise formulations of Artin's approximation theorem (due to Artin [1] in the algebraic case, and John Wavrik [17] in the analytic case) which assert that for every positive integer α there is a positive integer $\beta(\alpha)$ such that for each $\beta \geqslant \beta(\alpha)$, every β-order formal solution $\bar{y}(x)$ of $f(x, y) = 0$ (i.e. $\bar{y}(x)$ such that $f(x, \bar{y}(x)) \equiv 0 \bmod \mathfrak{m}^{\beta+1}$) may be approximated to order α by an algebraic or convergent solution. The method of our proof of Theorem B also provides invariant versions of these results. The one point worth noting is that for every positive integer γ, there exists a positive integer $\beta(\gamma)$ such that if $\bar{\eta}(u(x))$ is a $\beta(\gamma)$-order solution of

$$\sum_{j=1}^{t} h_j(u(x), \eta)\, G_j(x) = 0$$

(we are using the above notation), then there exist $\bar{\phi}^k(u)$, $k = 1, ..., m$, such that $(\bar{\eta}(u), \bar{\phi}(u))$ is a γ-order solution of

$$h_j(u, \eta) = \sum_{k=1}^{m} \phi^k\, h_j^k(u)\,, \quad 1 \leqslant j \leqslant t\,.$$

This statement follows from a simple extension of a theorem of Chevalley [14, 30.1].

4. A PROJECTION FORMULA

Let G be a compact Lie group and $M = L^2(G, dg)$ the space of complex-valued functions on G which are square integrable with respect to the normalized Haar measure dg. The mapping $f \to f^T$ from M into a space of continuous matrix-valued functions on G, defined for each irreducible complex representation T of G by the formula

$$\begin{aligned} f^T(h) &= \textstyle\int_G f(g^{-1}h)\, T(g)\, dg \\ &= T(h) \cdot \textstyle\int_G f(g^{-1})\, T(g)\, dg\,, \end{aligned}$$

where $h \in G$, is a generalized Fourier transform [10, Section 12] (cf. our proof of Theorem B in the complex analytic case). The Peter-Weyl theorem gives

$$f(h) = \sum_T \dim T \cdot \operatorname{tr} f^T(h)\,,$$

where the sum is taken over all finite dimensional inequivalent irreducible complex representations T. Moreover, the mapping $\pi_T : M \to M$ defined by

$$(\pi_T f)(h) = \dim T \cdot \operatorname{tr} f^T(h),$$

where $h \in G$, is the projection onto the largest invariant subspace of M whose irreducible invariant subspaces are all equivalent to the representation space of T.

Now let G be a reductive algebraic group defined over a field $\mathbf{k}$ of characteristic zero. A vector space M on which G acts linearly will be called a *G-module*. We will obtain projection formulas similar to the above in the following cases:

(a) M is a finite dimensional G-module;

(b) $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$;

(c) $M = \mathbf{k}\{x\}$, with $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$;

where, in cases (b) and (c), $x = (x_1, ..., x_n)$ denotes a coordinate system in a finite dimensional G-module V, and M has the induced action of G.

If L, M are G-modules, then the space $M^L = \operatorname{End}_{\mathbf{k}}(L, M)$ of $\mathbf{k}$-linear mappings $A : L \to M$ is a G-module, with the action of G defined by $g \cdot A = gAg^{-1}$. If L is an irreducible G-module, then $\mathbf{F}^L = \operatorname{End}_{\mathbf{k}}(L, L)^G$ is a field (in general not commutative). It is clear that $\mathbf{k}$ is a subfield of $\mathbf{F}^L$, and that the action of G on L commutes with the multiplication of elements of L by elements of $\mathbf{F}^L$.

We define a $\mathbf{k}$-homomorphism

$$J : \mathbf{F}^L \to \operatorname{End}_{\mathbf{k}}(\mathbf{F}^L, \mathbf{F}^L)$$

by

$$J(\lambda)(\mu) = \lambda \cdot \mu,$$

where λ, $\mu \in \mathbf{F}^L$, and let

$$\operatorname{tr}_L : \operatorname{End}_{\mathbf{k}}(L, L) \to \mathbf{k},$$

$$\operatorname{tr}_{\mathbf{F}^L} : \operatorname{End}_{\mathbf{k}}(\mathbf{F}^L, \mathbf{F}^L) \to \mathbf{k}$$

be the trace homomorphisms. It is not difficult to check that

$$\operatorname{tr}_L(\lambda) = m_L \operatorname{tr}_{\mathbf{F}^L}(J(\lambda))$$

for all $\lambda \in \mathbf{F}^L$, where m_L is the dimension of L over $\mathbf{F}^L$.

For each $v^* \in \mathrm{End}_{\mathbf{k}}(L, \mathbf{k})$ and $f \in M$, we denote by $v^* \otimes f \in \mathrm{End}_{\mathbf{k}}(L, M)$ the mapping $(v^* \otimes f)(w) = v^*(w) \cdot f$, $w \in L$. We also define a generalized trace homomorphism

$$\mathrm{Tr} : \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L) \to \mathrm{End}_{\mathbf{k}}(L, \mathbf{k})$$

by the formula

$$(\mathrm{Tr}\, v^{\#})(w) = \mathrm{tr}_{\mathbf{F}^L} J(v^{\#}(w)),$$

where $v^{\#} \in \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$ and $w \in L$.

In the following, E_M will denote a Reynolds operator for a G-module M; i.e. E_M is an invariant projection operator from M onto M^G [13, Definition 1.5].

PROPOSITION 4.1. *Suppose L is a finite dimensional irreducible G-module. Let $\{v_{j,L}\}_{1 \leq j \leq m_L}$ be a basis for L over $\mathbf{F}^L$, and $\{v^{\#}_{j,L}\}_{1 \leq j \leq m_L}$ be its dual basis. We consider one of the following G-modules M:*

(a) *M is a finite dimensional G-module;*

(b) $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$;

(c) $M = \mathbf{k}\{x\}$, $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$.

(*In the latter two cases, the action of G is induced by a linear action on the space of coordinates $x = (x_1, ..., x)$.*) *We define $\pi_L \in \mathrm{End}_{\mathbf{k}}(M, M)$ by*

$$\pi_L(f) = m_L \sum_{j=1}^{m_L} E_{M^L}(\mathrm{Tr}\, v^{\#}_{j,L} \otimes f)(v_{j,L}),$$

where $f \in M$. Then

(1) *π_L is a projection from M onto an invariant subspace whose irreducible invariant subspaces are all equivalent to L;*

(2) *for each $f \in M$.*

$$f = \sum_L \pi_L(f),$$

where the sum is taken over all finite dimensional inequivalent irreducible G-modules L (in cases (b) and (c) the sum converges in the Krull topology);

(3) *if I is an invariant ideal in M, in cases (b) and (c), then $\pi_L(f) \in I$ and*

$$E_{M^L}(\operatorname{Tr} v^{\#}_{j,L} \otimes f) \in \operatorname{End}_{\mathbf{k}}(L, I)$$

for all $f \in I$.

Remark 4.2. For each of the G-modules M of Proposition 4.1, there is a unique Reynolds operator E_{M^L}, and the mapping $M \to E_{M^L}$ is functorial. If M is finite dimensional, then this follows from the definition of "reductive". If $M = \mathbf{k}[x]$ or $\mathbf{k}[[x]]$ it follows from Cartier's lemma [13, p. 25]. If $M = \mathbf{C}\{x\}$ we define

$$E_{M^L}(f) = \int_H h \cdot f \, dh \,,$$

where $f \in M^L$ and H is a maximal compact subgroup of G. Finally if $M = \mathbf{R}\{x\}$, we put $E_{M^L}(f) = \operatorname{Re} E(f)$ for $f \in M^L$, where E is the Reynolds operator for the action of the complexification $G^{\mathbf{C}}$ of G on $\mathbf{C} \otimes_{\mathbf{R}} M^L$, and $\operatorname{Re} : \mathbf{C} \otimes_{\mathbf{R}} M^L \to M^L$ is the mapping $\operatorname{Re}(f) = \frac{1}{2}(f + \bar{f})$.

Remark 4.3. Proposition 4.1 provides an alternative proof of Theorem B when char $\mathbf{k} = 0$. Let I be the ideal in $\mathbf{k}[x]$ of an invariant algebraic subset of $\mathbf{k}^n$ (respectively the ideal in $\mathbf{k}\{x\}$ of a germ at 0 of an invariant analytic subset of $\mathbf{k}^n$, $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$). Then for each $f \in I$ and $v^{\#} \in \operatorname{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$, we define a polynomial mapping (respectively a germ at 0 of an analytic mapping)

$$F_{f, v^{\#}} : \mathbf{k}^n \to \operatorname{End}_{\mathbf{k}}(L, \mathbf{k})$$

by the formula

$$F_{f, v^{\#}}(x)(w) = (E_{M^L}(\operatorname{Tr} v^{\#} \otimes f)(w))(x) \,,$$

where $x \in \mathbf{k}^n$ and $w \in L$. Then $F_{f, v^{\#}}$ is equivariant and $X \subset F^{-1}_{f, v^{\#}}(0)$. We may now argue as in our proof of the algebraic case 2.3 of Theorem B. We use the facts that $(E_{M^L}(\operatorname{Tr} v^{\#}_{j,L} \otimes f)(v_{j,L}))(x)$ is a coordinate function of $F_{f, v^{\#}_{j,L}}$ and that $\Sigma_L \, \pi_L(f)$ converges to f in the Krull topology, to show that the ideal I coincides with the ideal in $\mathbf{k}[x]$ (respectively $\mathbf{k}\{x\}$) generated by the coordinate functions of all equivariant polynomial mappings (respectively germs at 0 of equivariant analytic mappings) F such that $X \subset F^{-1}(0)$.

Proof of Proposition 4.1. We first consider the case (a) that M is a finite dimensional G-module. We write M as a direct sum $M = \oplus_L M_L$ of G-submodules M_L, where the sum is taken over inequivalent irreducible G-submodules L, in such a way that each nonzero irreducible G-submodule of M_L is equivalent to L. Let $f = \Sigma_L f_L$, where $f_L \in M_L$. It is enough to prove that $\pi_L f = f_L$; in other words that $\pi_L f_{L'} = 0$ if $L \neq L'$, and $\pi_L f_L = f_L$.

The first condition follows from the fact that $\mathrm{End}_{\mathbf{k}}(L, L')^G = 0$. Using the functorial property of the Reynolds operators, we reduce the second to the case $M = L$; i.e. we must prove $\pi_L f = f$ for all $f \in L$. Since

$$f = \sum_{j=1}^{m_L} v^{\#}_{j,L}(f) \cdot v_{j,L},$$

it is enough to show that

$$m_L \cdot E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) = v^{\#}(f)$$

for all $f \in L$ and $v^{\#} \in \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$.

For each $\beta \in \mathbf{F}^L$, we define a homomorphism

$$\mathrm{tr}_\beta : \mathrm{End}_{\mathbf{k}}(L, L) \to \mathbf{k}$$

by the formula $\mathrm{tr}_\beta(A) = \mathrm{tr}_L(\beta \cdot A)$, $A \in \mathrm{End}_{\mathbf{k}}(L, L)$. Then tr_β is G-invariant, so that

$$\mathrm{tr}_\beta \circ E_{LL} = \mathrm{tr}_\beta .$$

By a direct computation, we also check that

$$\mathrm{tr}_\beta(\mathrm{Tr}\, v^{\#} \otimes f) = \mathrm{tr}_{\mathbf{F}^L} J\left(v^{\#}(\beta \cdot f)\right).$$

Hence for each $\beta \in \mathbf{F}^L$,

$$\begin{aligned}\mathrm{tr}_\beta\left(m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f)\right) &= m_L\, \mathrm{tr}_{\mathbf{F}^L} J\left(v^{\#}(\beta \cdot f)\right)\\ &= \mathrm{tr}_\beta v^{\#}(f).\end{aligned}$$

This implies that

$$m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) = v^{\#}(f),$$

because otherwise, letting β be the reciprocal of $m_L E_{LL}(\mathrm{Tr}\, v^{\#} \otimes f) - v^{\#}(f)$ in $\mathbf{F}^L$, we would have $\dim_{\mathbf{k}} L = \mathrm{tr}_L(\mathrm{id}) = 0$, contradicting char $\mathrm{k} = 0$. This completes the proof of Proposition 4.1 in the case (a).

In the case $M = \mathbf{k}[x]$, it follows from the functorial property of the Reynolds operators that $\pi_L(\mathbf{k}[x]_c) \subset \mathbf{k}[x]_c$ for all $c \in \mathbf{N}$. Hence properties (1) and (2) of Proposition 4.1 follow from the finite dimensional case (a). Moreover, if I is an invariant ideal in $\mathbf{k}[x]$, then $I \cap \mathbf{k}[x]_c$ is an invariant subspace of $\mathbf{k}[x]_c$, and

$$I = \bigcup_{c \in \mathbf{N}} I \cap \mathbf{k}[x]_c .$$

Therefore $\pi_L f \in I$ and

$$E_{M^L}(\mathrm{Tr}\, v_j^{\#}{}_{,L} \otimes f) \in \mathrm{End}_{\mathbf{k}}(L, I)$$

as required in property (c).

It remains to consider the cases $M = \mathbf{k}[[x]]$, and $M = \mathbf{k}\{x\}$ with $\mathbf{k} = \mathbf{R}, \mathbf{C}$. In each case let $\mathfrak{m}$ be the maximal ideal and let M_c, $c \in \mathbf{N}$, be the invariant subspace of M of polynomials of degree at most c. If $f \in \mathfrak{m}^c$, then $\mathrm{Tr}\, v^{\#} \otimes f \in \mathrm{End}_{\mathbf{k}}(L, \mathfrak{m}^c)$ for all $v^{\#} \in \mathrm{End}_{\mathbf{F}^L}(L, \mathbf{F}^L)$, so that $\pi_L f \in \mathfrak{m}^c$. Likewise if $f \in M_c$ then $\pi_L f \in M_c$. For each $f \in M$ and $c \in \mathbf{N}$, we write

$$f = T^c f + R^c f ,$$

where $T^c f \in M_c$ and $R^c f \in \mathfrak{m}^{c+1}$. Then for all $f \in M$ and $c \in \mathbf{N}$,

$$\pi_L^2 f - \pi_L f = \pi_L^2(R^c f) - \pi_L(R^c f) \in \mathfrak{m}^{c+1} ,$$

so that $\pi_L^2 = \pi_L$.

For each $c \in \mathbf{N}$, let P_c be the natural projection from M to its subspace of homogeneous polynomials of degree c. Each $f \in M$ may be written $f = \Sigma_c P_c f$. Then $\pi_L \circ P_c = P_c \circ \pi_L$ for every $c \in \mathbf{N}$ and every irreducible G-module L. Suppose that N is a nonzero irreducible G-submodule of $\pi_L(M)$. Then either $P_c(N) = 0$ or $P_c : N \to P_c(N)$ is an equivalence of G-modules. Choose $c \in \mathbf{N}$ such that $P_c(N) \neq 0$. Then N is equivalent to $P_c(N)$ and $P_c(N) = \pi_L(P_c(N)) \subset \pi_L(M_c)$ is equivalent to L, by the finite dimensional case (a). This completes the proof of property (1) for $M = \mathbf{k}[[x]]$ or $\mathbf{k}\{x\}$.

To obtain property (2), we let $N(-1) = \varnothing$ and let $N(c)$, $c \in \mathbf{N}$, be the set of all inequivalent irreducible G-modules appearing in the decomposition of M_c as a direct sum of irreducible G-modules. Then for each $c \in \mathbf{N}$,

$$f - \sum_{L \in N(c)} \pi_L f = R^c f - \sum_{L \in N(c)} \pi_L R^c f \in \mathfrak{m}^{c+1} .$$

Since $\pi_L f = 0$ if $L \notin \cup_c N(c)$, then $\Sigma_L \pi_L f$ converges to f in the Krull topology.

We finally consider property (3) for $M = \mathbf{k}[[x]]$ or $\mathbf{k}\{x\}$. Let I be an invariant ideal in M. Then $I \cap M_c$ is an invariant subspace of M_c. It follows that if $f \in I$, then $\pi_L f \in I + \mathfrak{m}^{c+1}$ for all $c \in \mathbf{N}$, so that $\pi_L f \in I$ by Krull's theorem [14, 16.7]. Moreover

$$\operatorname{End}_{\mathbf{k}}(L, I) = \bigcap_{c \in \mathbf{N}} \operatorname{End}_{\mathbf{k}}(L, I + \mathfrak{m}^{c+1}).$$

Let $f \in I$. Writing $f = T^c f + R^c f$ and using the functorial property of the Reynolds operators, we have

$$E_{M^L}(\operatorname{Tr} v_{j,L}^{\#} \otimes T^c f) \in \operatorname{End}_{\mathbf{k}}(L, I \cap M_c),$$

$$E_{M^L}(\operatorname{Tr} v_{j,L}^{\#} \otimes R^c f) \in \operatorname{End}_{\mathbf{k}}(L, \mathfrak{m}^{c+1})$$

for all $c \in \mathbf{N}$. Since $I + \mathfrak{m}^{c+1} = I \cap M_c + \mathfrak{m}^{c+1}$, it follows that

$$E_{M^L}(\operatorname{Tr} v_{j,L}^{\#} \otimes f) \in \operatorname{End}_{\mathbf{k}}(L, I).$$

This completes the proof of Proposition 4.1.

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