Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 25 (1979)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

Autor: Bierstone, Edward / Milman, Pierre

Kapitel: 3. Proof of Theorem A

DOI: https://doi.org/10.5169/seals-50374

Suppose that G acts linearly on $V = \mathbf{R}^n$, and that X is a germ at 0 of an invariant real analytic subset of V. The complexification $X^{\mathbf{C}}$ of X is a germ at 0 of a complex analytic subset of $V^{\mathbf{C}} = \mathbf{C}^n$. The complexification $X^{\mathbf{C}}$ is invariant under the induced action of $G^{\mathbf{C}}$ on $V^{\mathbf{C}}$.

By the complex analytic case 2.1, there is a linear action of $G^{\mathbf{C}}$ on a finite dimensional complex vector space W, and a germ H at 0 of a $G^{\mathbf{C}}$-equivariant holomorphic mapping of some neighborhood of $0 \in V^{\mathbf{C}}$ into W, such that $X^{\mathbf{C}} = H^{-1}(0)$.

Let Y be W with its underlying real structure. Then $F = H \mid V : V \to Y$ is G-equivariant, and $X = F^{-1}(0)$.

2.3. The algebraic case. Our ground field $\mathbf{k}$ is now arbitrary. Let G be a reductive algebraic group acting linearly on $V = \mathbf{k}^n$, and let X be an invariant algebraic subset of V. Let I be the ideal in $\mathbf{k}[x]$ of polynomials which vanish on X, and $\mathbf{k}[x]_c$ be the subspace of $\mathbf{k}[x]$ of polynomials of degree at most c. Then I and $\mathbf{k}[x]_c$ are invariant subsets of $\mathbf{k}[x]$.

For each $c \in \mathbf{N}$, we define a polynomial mapping

$$F_c : V \to \mathrm{End}_{\mathbf{k}}\left(I \cap \mathbf{k}[x]_c, \mathbf{k}\right)$$

by the formula $F_c(x)(h) = h(x)$, where $x \in V$ and $h \in I \cap \mathbf{k}[x]_c$. Then F_c is equivariant and $X \subset F_c^{-1}(0)$ for all $c \in \mathbf{N}$.

We consider the ideal J in $\mathbf{k}[x]$ generated by the coordinate functions of all equivariant polynomial mappings defined on V, which vanish on X. Since J is finitely generated, it suffices to show that $J = I$. Clearly $J \subset I$. On the other hand, suppose $h \in I \cap \mathbf{k}[x]_c$, $h \neq 0$. Let $\{e_j\}_{1 \leq j \leq q}$ be a basis of the vector space $I \cap \mathbf{k}[x]_c$, such that $e_1 = h$. Then h is the first coordinate function of the equivariant mapping F_c, with respect to the dual basis $\{e_j^*\}_{1 \leq j \leq q}$ in $\mathrm{End}_{\mathbf{k}}\left(I \cap \mathbf{k}[x]_c, \mathbf{k}\right)$. Since $X \subset F_c^{-1}(0)$, then $h \in J$. Hence $J = I$ as required.

This case of Theorem B may also be obtained from a lemma of Cartier [13, p. 25].

3. Proof of Theorem A

The formal power series $\bar{y}(x) \in \mathbf{k}[[x]]^p$ define a local $\mathbf{k}$-homomorphism $\phi : \mathbf{k}\{x, y\} \to \mathbf{k}[[x]]$ (or a $\mathbf{k}$-homomorphism $\phi : \mathbf{k}[x, y] \to \mathbf{k}[[x]]$ in the algebraic case) by substitution: $h(x, y) \to h\left(x, \bar{y}(x)\right)$.

Let X be the germ at 0 in $V \times W = \mathbf{k}^{n+p}$ of a closed analytic subset (or the closed algebraic subset of $V \times W$ in the algebraic case) defined by the prime ideal ker ϕ. It follows from Artin's approximation theorem that ker ϕ satisfies the nullstellensatz (whether or not $\mathbf{k}$ is algebraically closed). In other words, if $h(x, y)$ vanishes on X, then $h(x, y) \in \ker \phi$. In fact if h vanishes on X, then for any $c \in \mathbf{N}$ we can find a convergent series solution $y(x)$ of the system of equations determined by the ideal ker ϕ, such that $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$. Then $h\left(x, y(x)\right) = 0$ and $h\left(x, y(x)\right) \equiv h\left(x, \bar{y}(x)\right) \bmod \mathfrak{m}^c$. Hence $h \in \ker \phi$.

It follows that Theorem B reduces Theorem A to the case of an equivariant equation. We may assume that $f(x, y) \in \mathbf{k}\{x, y\}^q$ (respectively $f(x, y) \in \mathbf{k}[x, y]^q$) is the germ of an equivariant analytic mapping (respectively the equivariant polynomial mapping) given by Theorem B for the invariant analytic set germ (respectively algebraic set) X.

From now on, then, we assume that G acts linearly on $V = \mathbf{k}^n$, $W = \mathbf{k}^p$ and $Y = \mathbf{k}^q$, and that $f(x, y)$ is a germ of an equivariant analytic mapping (or an equivariant polynomial mapping in the algebraic case).

Since G is reductive, then $\mathbf{k}[x]^G$ is finitely generated (as a $\mathbf{k}$-algebra) by homogeneous polynomials $u_1(x), \ldots, u_r(x) \in \mathbf{k}[x]^G$ [13, Theorem 1.1]. Hence the homomorphisms

$$u^*: \mathbf{k}[u] \to \mathbf{k}[x]^G,$$
$$\hat{u}^*: \mathbf{k}[[u]] \to \mathbf{k}[[x]]^G$$

defined by substitution $h(u_1, \ldots, u_r) \to h\left(u_1(x), \ldots, u_r(x)\right)$ are surjective. If $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, then the induced homomorphism

$$\tilde{u}^*: \mathbf{k}\{u\} \to \mathbf{k}\{x\}^G$$

is surjective by a result of Luna [11].

In the remainder of the proof we consider only the analytic case. The proof of the algebraic case is identical, if we replace the analytic version of Artin's approximation theorem by the algebraic version.

Remark 3.1. If G acts trivially on Y, i.e. $f_j(x, y) \in \mathbf{k}\{x, y\}^G$, $j = 1, \ldots, q$, then our theorem follows immediately. In fact let $I = \ker u^*$. Then $\ker \hat{u}^* = I \cdot \mathbf{k}[[u]]$ and $\ker \tilde{u}^* = I \cdot \mathbf{k}\{u\}$ (the former equality follows by expressing a power series in $\mathbf{k}[[u]]$ as a sum of weighted homogeneous polynomials, weighted by the degrees of the u_i, and the latter then by Artin's theorem). Suppose that $F_1(x), \ldots, F_s(x)$ generate the module of equivariant polynomial mappings of V into W over the ring $\mathbf{k}[x]^G$ of invariant polynomials on V. Since $f(x, \Sigma_{i=1}^s \eta_i F_i(x))$ is invariant in (x, η), where G acts trivially on the variables $\eta = (\eta_1, \ldots, \eta_s)$, then there exists $h \in \mathbf{k}[u, \eta]^q$, such that

$$f\left(x, \sum_{i=1}^{s} \eta_i F_i(x)\right) = h(u(x), \eta)\,.$$

If $\bar{y}(x) = \sum_{i=1}^{s} \bar{\eta}_i(u(x)) F_i(x)$ is a formal solution of $f(x, y) = 0$, then

$$h(u, \bar{\eta}(u)) \in I \cdot \mathbf{k}[[u]]^q\,.$$

By Artin's theorem we may approximate $\bar{\eta}(u)$ by a convergent $\eta(u)$ such that

$$h(u, \eta(u)) \in I \cdot \mathbf{k}\{u\}^q\,.$$

Then

$$y(x) = \sum_{i=1}^{s} \eta_i(u(x)) F_i(x)$$

is an analytic solution of $f(x, y) = 0$, approximating $\bar{y}(x)$.

In general, suppose that $F_1(x), \ldots, F_s(x)$ (respectively $G_1(x), \ldots, G_t(x)$) generate the module of equivariant polynomial mappings of V into W (respectively of V into Y), over the ring $\mathbf{k}[x]^G$. Then we may write

$$f\left(x, \sum_{i=1}^{s} \eta_i F_i(x)\right) = \sum_{j=1}^{t} h_j(u(x), \eta)\, G_j(x)\,,$$

where $h_j(u, \eta) \in \mathbf{k}\{u, \eta\}$, $j = 1, \ldots, t$. (This may be proved, for example, in the same way as Proposition 3.2 of [5]).

Let M (respectively $\hat{M}$) be the $\mathbf{k}[u]$ − (respectively $\mathbf{k}[[u]]$ −) submodule of $\mathbf{k}[u]^t$ (respectively $\mathbf{k}[[u]]^t$) of t-tuples $(h_1(u), \ldots, h_t(u))$ such that

$$\sum_{j=1}^{t} h_j(u(x))\, G_j(x) = 0\,.$$

Suppose that M is generated by $h^k(u) = (h_1^k(u), \ldots, h_t^k(u))$, $k = 1, \ldots, m$. Then $\hat{M} = \mathbf{k}[[u]] \cdot M$. To see this, we may assume that $G_j(x)$ is homogeneous, of degree d_j say. Let $h(u) = (h_1(u), \ldots, h_t(u)) \in \hat{M}$. We write

$$h_j(u) = \sum_{l} h_{jl}(u)\,,$$

where h_{jl} is weighted homogeneous (weighted by the degrees of the polynomials $u_i(x)$) of degree $l - d_j$. Then

$$\sum_{j=1}^{t} h_{jl}\big(u(x)\big)\, G_j(x) = 0$$

for each l; i.e. $\big(h_{1l}(u), \ldots, h_{tl}(u)\big) \in M$. Hence we may write

$$\big(h_{1l}(u), \ldots, h_{tl}(u)\big) = \sum_{k=1}^{m} \phi_l^k(u)\, h^k(u),$$

where $\phi_l^k(u) \in \mathbf{k}[u]$, so that

$$h(u) = \sum_{k=1}^{m} \Big(\sum_{l} \phi_l^k(u)\Big) h^k(u)$$

as required.

Now suppose that $\bar{y}(x) = \Sigma_{i=1}^{s} \bar{\eta}_i\big(u(x)\big) F_i(x)$ is a formal solution of $f(x, y) = 0$; i.e.

$$\Big(h_1\big(u, \bar{\eta}(u)\big), \ldots, h_t\big(u, \bar{\eta}(u)\big)\Big) \in \hat{M},$$

or

$$h_j\big(u, \bar{\eta}(u)\big) = \sum_{k=1}^{m} \bar{\phi}^k(u)\, h_j^k(u), \quad 1 \leqslant j \leqslant t,$$

where $\bar{\phi}^k(u) \in \mathbf{k}[[u]]$, $1 \leqslant k \leqslant m$. Then by Artin's theorem there are convergent power series $\eta(u)$, $\phi^k(u)$, such that

$$h_j\big(u, \eta(u)\big) = \sum_{k=1}^{m} \phi^k(u)\, h_j^k(u), \quad 1 \leqslant j \leqslant t,$$

and $\eta(u) \equiv \bar{\eta}(u)$, $\phi^k(u) \equiv \bar{\phi}^k(u) \bmod m^c$. Let

$$y(x) = \sum_{i=1}^{s} \eta_i\big(u(x)\big) F_i(x).$$

Then $y(x)$ is equivariant, $y(x) \equiv \bar{y}(x) \bmod m^c$, and

$$f\Big(x, y(x)\Big) = f\Big(x, \sum_{i=1}^{s} \eta_i\big(u(x)\big) F_i(x)\Big)$$

$$= \sum_{j=1}^{t} h_j\Big(u(x), \eta\big(u(x)\big)\Big) G_j(x) = 0.$$

Remark 3.2. There are more precise formulations of Artin's approximation theorem (due to Artin [1] in the algebraic case, and John Wavrik [17] in the analytic case) which assert that for every positive integer α there is a positive integer $\beta(\alpha)$ such that for each $\beta \geqslant \beta(\alpha)$, every β-order formal solution $\bar{y}(x)$ of $f(x, y) = 0$ (i.e. $\bar{y}(x)$ such that $f(x, \bar{y}(x)) \equiv 0 \bmod \mathfrak{m}^{\beta+1}$) may be approximated to order α by an algebraic or convergent solution. The method of our proof of Theorem B also provides invariant versions of these results. The one point worth noting is that for every positive integer γ, there exists a positive integer $\beta(\gamma)$ such that if $\bar{\eta}(u(x))$ is a $\beta(\gamma)$-order solution of

$$\sum_{j=1}^{t} h_j(u(x), \eta) \, G_j(x) = 0$$

(we are using the above notation), then there exist $\bar{\phi}^k(u)$, $k = 1, \ldots, m$, such that $(\bar{\eta}(u), \bar{\phi}(u))$ is a γ-order solution of

$$h_j(u, \eta) = \sum_{k=1}^{m} \phi^k \, h_j^k(u), \quad 1 \leqslant j \leqslant t.$$

This statement follows from a simple extension of a theorem of Chevalley [14, 30.1].

4. A projection formula

Let G be a compact Lie group and $M = L^2(G, dg)$ the space of complex-valued functions on G which are square integrable with respect to the normalized Haar measure dg. The mapping $f \to f^T$ from M into a space of continuous matrix-valued functions on G, defined for each irreducible complex representation T of G by the formula

$$\begin{aligned} f^T(h) &= \int_G f(g^{-1}h) \, T(g) \, dg \\ &= T(h) \cdot \int_G f(g^{-1}) \, T(g) \, dg, \end{aligned}$$

where $h \in G$, is a generalized Fourier transform [10, Section 12] (cf. our proof of Theorem B in the complex analytic case). The Peter-Weyl theorem gives

$$f(h) = \sum_T \dim T \cdot \operatorname{tr} f^T(h),$$