

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 25 (1979)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS  
**Autor:** Bierstone, Edward / Milman, Pierre  
**Kapitel:** 2. Proof of Theorem B  
**DOI:** <https://doi.org/10.5169/seals-50374>

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## 2. PROOF OF THEOREM B

2.1. The complex analytic case. Let  $G$  be a reductive complex algebraic group. Then  $G$  is the universal complexification of a compact real Lie group  $G^{\mathbf{R}}$  [9], [8, XVII.5].

Suppose that  $G$  acts linearly on  $V = \mathbf{C}^n$ , and that  $X$  is a germ at 0 of an invariant closed analytic subset of  $V$ . Let  $I$  be the ideal in  $\mathbf{C}\{x\} = \mathbf{C}\{x_1, \dots, x_n\}$  of germs of holomorphic functions which vanish on  $X$ . Suppose that  $I$  is generated by  $f_1, \dots, f_k$ .

For any irreducible complex representation  $T: G \rightarrow GL(W)$  of  $G$ , we consider the action of  $G$  on the space  $\text{End}_{\mathbf{C}}(W, W)$  of complex linear endomorphisms defined by

$$(g \cdot \lambda)(w) = T(g) \lambda(w),$$

where  $g \in G$ ,  $w \in W$  and  $\lambda \in \text{End}_{\mathbf{C}}(W, W)$ . For each  $i = 1, \dots, k$ , we consider the mapping

$$f_i^T: V \rightarrow \text{End}_{\mathbf{C}}(W, W)$$

$$f_i^T(x) = \int_{G^{\mathbf{R}}} f_i(g^{-1}x) T(g) dg,$$

defined in an open neighborhood of 0 where  $f_i$  converges. Then  $f_i^T$  is equivariant with respect to the actions of  $G^{\mathbf{R}}$  on  $V$  and  $\text{End}_{\mathbf{C}}(W, W)$ , and hence with respect to the actions of  $G$  (the “unitarian trick”). Furthermore  $f_i(gx) = 0$  for all  $g \in G$  if and only if  $f_i^T(x) = 0$  for all irreducible complex representations  $T$  of  $G^{\mathbf{R}}$  (cf. [10, 12.2]; this is essentially the Peter-Weyl theorem).

Hence  $X$  is defined by the equations

$$f_i^T(x) = 0,$$

where  $1 \leq i \leq k$  and  $T$  runs over all irreducible complex representations of  $G^{\mathbf{R}}$ . It follows that  $X$  is defined by a finite subset of these equivariant equations.

2.2. The real analytic case. Let  $G$  be a reductive real algebraic group. Then the universal complexification  $G^{\mathbf{C}}$  of  $G$  is a reductive complex algebraic group [8, XVIII.4].

Suppose that  $G$  acts linearly on  $V = \mathbf{R}^n$ , and that  $X$  is a germ at 0 of an invariant real analytic subset of  $V$ . The complexification  $X^{\mathbf{C}}$  of  $X$  is a germ at 0 of a complex analytic subset of  $V^{\mathbf{C}} = \mathbf{C}^n$ . The complexification  $X^{\mathbf{C}}$  is invariant under the induced action of  $G^{\mathbf{C}}$  on  $V^{\mathbf{C}}$ .

By the complex analytic case 2.1, there is a linear action of  $G^{\mathbf{C}}$  on a finite dimensional complex vector space  $W$ , and a germ  $H$  at 0 of a  $G^{\mathbf{C}}$ -equivariant holomorphic mapping of some neighborhood of  $0 \in V^{\mathbf{C}}$  into  $W$ , such that  $X^{\mathbf{C}} = H^{-1}(0)$ .

Let  $Y$  be  $W$  with its underlying real structure. Then  $F = H|_V: V \rightarrow Y$  is  $G$ -equivariant, and  $X = F^{-1}(0)$ .

2.3. The algebraic case. Our ground field  $\mathbf{k}$  is now arbitrary. Let  $G$  be a reductive algebraic group acting linearly on  $V = \mathbf{k}^n$ , and let  $X$  be an invariant algebraic subset of  $V$ . Let  $I$  be the ideal in  $\mathbf{k}[x]$  of polynomials which vanish on  $X$ , and  $\mathbf{k}[x]_c$  be the subspace of  $\mathbf{k}[x]$  of polynomials of degree at most  $c$ . Then  $I$  and  $\mathbf{k}[x]_c$  are invariant subsets of  $\mathbf{k}[x]$ .

For each  $c \in \mathbf{N}$ , we define a polynomial mapping

$$F_c: V \rightarrow \text{End}_{\mathbf{k}}(I \cap \mathbf{k}[x]_c, \mathbf{k})$$

by the formula  $F_c(x)(h) = h(x)$ , where  $x \in V$  and  $h \in I \cap \mathbf{k}[x]_c$ . Then  $F_c$  is equivariant and  $X \subset F_c^{-1}(0)$  for all  $c \in \mathbf{N}$ .

We consider the ideal  $J$  in  $\mathbf{k}[x]$  generated by the coordinate functions of all equivariant polynomial mappings defined on  $V$ , which vanish on  $X$ . Since  $J$  is finitely generated, it suffices to show that  $J = I$ . Clearly  $J \subset I$ . On the other hand, suppose  $h \in I \cap \mathbf{k}[x]_c$ ,  $h \neq 0$ . Let  $\{e_j\}_{1 \leq j \leq q}$  be a basis of the vector space  $I \cap \mathbf{k}[x]_c$ , such that  $e_1 = h$ . Then  $h$  is the first coordinate function of the equivariant mapping  $F_c$ , with respect to the dual basis  $\{e_j^*\}_{1 \leq j \leq q}$  in  $\text{End}_{\mathbf{k}}(I \cap \mathbf{k}[x]_c, \mathbf{k})$ . Since  $X \subset F_c^{-1}(0)$ , then  $h \in J$ . Hence  $J = I$  as required.

This case of Theorem B may also be obtained from a lemma of Cartier [13, p. 25].

### 3. PROOF OF THEOREM A

The formal power series  $\bar{y}(x) \in \mathbf{k}[[x]]^p$  define a local  $\mathbf{k}$ -homomorphism  $\phi: \mathbf{k}\{x, y\} \rightarrow \mathbf{k}[[x]]$  (or a  $\mathbf{k}$ -homomorphism  $\phi: \mathbf{k}[x, y] \rightarrow \mathbf{k}[[x]]$  in the algebraic case) by substitution:  $h(x, y) \rightarrow h(x, \bar{y}(x))$ .