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2. Proof of Theorem B

2.1. The complex analytic case. Let G be a reductive complex algebraic group. Then G is the universal complexification of a compact real Lie group $G^{\mathbf{R}}$ [9], [8, XVII.5].

Suppose that G acts linearly on $V = \mathbf{C}^n$, and that X is a germ at 0 of an invariant closed analytic subset of V. Let I be the ideal in $\mathbf{C}\{x\} = \mathbf{C}\{x_1, \ldots, x_n\}$ of germs of holomorphic functions which vanish on X. Suppose that I is generated by $f_1, \ldots, f_k$.

For any irreducible complex representation $T\colon G \to GL(W)$ of G, we consider the action of G on the space $\mathrm{End}_{\mathbf{C}}(W, W)$ of complex linear endomorphisms defined by

$$(g \cdot \lambda)(w) = T(g)\,\lambda(w),$$

where $g \in G$, $w \in W$ and $\lambda \in \mathrm{End}_{\mathbf{C}}(W, W)$. For each $i = 1, \ldots, k$, we consider the mapping

$$f_i^T\colon V \to \mathrm{End}_{\mathbf{C}}(W, W)$$

$$f_i^T(x) = \int_{G^{\mathbf{R}}} f_i(g^{-1}x)\, T(g)\, dg\,,$$

defined in an open neighborhood of 0 where f_i converges. Then f_i^T is equivariant with respect to the actions of $G^{\mathbf{R}}$ on V and $\mathrm{End}_{\mathbf{C}}(W, W)$, and hence with respect to the actions of G (the "unitarian trick"). Furthermore $f_i(gx) = 0$ for all $g \in G$ if and only if $f_i^T(x) = 0$ for all irreducible complex representations T of $G^{\mathbf{R}}$ (cf. [10, 12.2]; this is essentially the Peter-Weyl theorem).

Hence X is defined by the equations

$$f_i^T(x) = 0\,,$$

where $1 \leqslant i \leqslant k$ and T runs over all irreducible complex representations of $G^{\mathbf{R}}$. It follows that X is defined by a finite subset of these equivariant equations.

2.2. The real analytic case. Let G be a reductive real algebraic group. Then the universal complexification $G^{\mathbf{C}}$ of G is a reductive complex algebraic group [8, XVIII.4].

Suppose that G acts linearly on $V = \mathbf{R}^n$, and that X is a germ at 0 of an invariant real analytic subset of V. The complexification $X^{\mathbf{C}}$ of X is a germ at 0 of a complex analytic subset of $V^{\mathbf{C}} = \mathbf{C}^n$. The complexification $X^{\mathbf{C}}$ is invariant under the induced action of $G^{\mathbf{C}}$ on $V^{\mathbf{C}}$.

By the complex analytic case 2.1, there is a linear action of $G^{\mathbf{C}}$ on a finite dimensional complex vector space W, and a germ H at 0 of a $G^{\mathbf{C}}$-equivariant holomorphic mapping of some neighborhood of $0 \in V^{\mathbf{C}}$ into W, such that $X^{\mathbf{C}} = H^{-1}(0)$.

Let Y be W with its underlying real structure. Then $F = H \mid V : V \to Y$ is G-equivariant, and $X = F^{-1}(0)$.

2.3. The algebraic case. Our ground field $\mathbf{k}$ is now arbitrary. Let G be a reductive algebraic group acting linearly on $V = \mathbf{k}^n$, and let X be an invariant algebraic subset of V. Let I be the ideal in $\mathbf{k}[x]$ of polynomials which vanish on X, and $\mathbf{k}[x]_c$ be the subspace of $\mathbf{k}[x]$ of polynomials of degree at most c. Then I and $\mathbf{k}[x]_c$ are invariant subsets of $\mathbf{k}[x]$.

For each $c \in \mathbf{N}$, we define a polynomial mapping

$$F_c: V \to \mathrm{End}_{\mathbf{k}}\left(I \cap \mathbf{k}[x]_c, \mathbf{k}\right)$$

by the formula $F_c(x)(h) = h(x)$, where $x \in V$ and $h \in I \cap \mathbf{k}[x]_c$. Then F_c is equivariant and $X \subset F_c^{-1}(0)$ for all $c \in \mathbf{N}$.

We consider the ideal J in $\mathbf{k}[x]$ generated by the coordinate functions of all equivariant polynomial mappings defined on V, which vanish on X. Since J is finitely generated, it suffices to show that $J = I$. Clearly $J \subset I$. On the other hand, suppose $h \in I \cap \mathbf{k}[x]_c$, $h \neq 0$. Let $\{e_j\}_{1 \leq j \leq q}$ be a basis of the vector space $I \cap \mathbf{k}[x]_c$, such that $e_1 = h$. Then h is the first coordinate function of the equivariant mapping F_c, with respect to the dual basis $\{e_j^*\}_{1 \leq j \leq q}$ in $\mathrm{End}_{\mathbf{k}}\left(I \cap \mathbf{k}[x]_c, \mathbf{k}\right)$. Since $X \subset F_c^{-1}(0)$, then $h \in J$. Hence $J = I$ as required.

This case of Theorem B may also be obtained from a lemma of Cartier [13, p. 25].

3. Proof of Theorem A

The formal power series $\bar{y}(x) \in \mathbf{k}[[x]]^p$ define a local $\mathbf{k}$-homomorphism $\phi: \mathbf{k}\{x, y\} \to \mathbf{k}[[x]]$ (or a $\mathbf{k}$-homomorphism $\phi: \mathbf{k}[x, y] \to \mathbf{k}[[x]]$ in the algebraic case) by substitution: $h(x, y) \to h\left(x, \bar{y}(x)\right)$.