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# INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

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## 1. INTRODUCTION

Let **k** be a field of characteristic zero with a non-trivial valuation.

We consider a system of analytic equations

$$(*) \qquad f(x, y) = 0 ,$$

where

$$f(x, y) = \big(f_1(x, y), \ldots, f_q(x, y)\big)$$

are convergent series in the variables

$$x = (x_1, \ldots, x_n) ,$$
$$y = (y_1, \ldots, y_p) .$$

Suppose that

$$\bar{y}(x) = \big(\bar{y}_1(x), \ldots, \bar{y}_p(x)\big) , \quad \bar{y}_j(x) \in \mathbf{k}[[x]] ,$$

are formal power series without constant term which solve (*); i.e. such that $f(x, \bar{y}(x)) = 0$. Let $c$ be a non-negative integer. Artin's approximation theorem [3] asserts that there exists a convergent series solution

$$y(x) = \big(y_1(x), \ldots, y_p(x)\big) , \quad y_j(x) \in \mathbf{k}\{x\} ,$$

of (*), such that

$$y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c .$$

Here $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{k}[[x]]$.

Artin also proved an algebraic analogue of this theorem [1]. It says that if $f(x, y) = 0$ is a system of polynomial equations with formal series solution $\bar{y}(x)$, then a series solution $y(x)$ may be found such that the $y_j(x)$ are algebraically dependent on $x_1, \ldots, x_n$ (we will say that the $y_j(x)$ are "algebraic"; cf. [2]). In this analogue **k** is an arbitrary field.

Let $G$ be a reductive algebraic group (i.e. $G$ is linear and every rational representation of $G$ is completely reducible). Suppose that $G$ acts linearly on $V = \mathbf{k}^n$ and $W = \mathbf{k}^p$. We will say that $\bar{y}(x) \in \mathbf{k}[[x]]^p$ is *equivariant* if

$$\bar{y}(gx) = g\bar{y}(x), \quad g \in G.$$

We will prove the following theorem.

THEOREM A. *Suppose* $\mathbf{k} = \mathbf{R}$ *or* $\mathbf{C}$, *and that* $\bar{y}(x) \in \mathbf{k}[[x]]^p$ *is an equivariant formal power series solution of* (*), $\bar{y}(0) = 0$. *Let* $c \in \mathbf{N}$. *Then there exists an equivariant convergent series solution* $y(x)$ *of* (*), *such that* $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$.

*Moreover, if* $f(x, y) = 0$ *is a system of polynomial equations (where* $\mathbf{k}$ *is any field), then there exists an equivariant algebraic solution* $y(x)$, *such that* $y(x) \equiv \bar{y}(x) \bmod \mathfrak{m}^c$.

*Remark 1.1.* Theorem A may be regarded in the context of the question: What properties of a formal solution of (*) may be preserved in an analytic solution? Artin [2] asked whether there is a convergent solution such that some of the variables $x_i$ are missing in some of the series $y_j(x)$, provided there is a formal solution with the same property. Gabrielov [6] answered this question negatively (see also [4]). In [12] it is shown that if a formal solution of a system of real analytic equations satisfies the Cauchy-Riemann equations, then it may be approximated by complex analytic solutions.

*Remark 1.2.* Suppose that $\pi(x) \in \mathbf{C}\{x\}^r$ is an analytically regular germ of an analytic mapping (terminology of Gabrielov [7]). Let $F_i(x) \in \mathbf{C}\{x\}^p$, $i = 1, \ldots, q$. We may ask whether formal relations among the $F_i$ of the form $(h_1(\pi(x)), \ldots, h_q(\pi(x)))$; i.e. $q$-tuples of formal power series of this form such that

$$\sum_{i=1}^{q} h_i(\pi(x)) F_i(x) = 0,$$

are generated by analytic relations of the same form. This question generalizes Gabrielov's problem in [7]. The answer is *no* in general, but the method of our proof of Theorem A shows it is *yes* if $\pi$ is a finite analytic germ. As in our proof of Theorem A, it is then easy to see that a formal solution $\bar{y}(\pi(x))$ of a system of complex analytic equations $f(x, y) = 0$ may be approximated by analytic solutions of the same form. We are grateful to Joseph Becker for pointing out the latter result to us.

*Remark 1.3.* Tougeron [16] has proved a generalization of Artin's theorem which asserts, in particular, that every formal solution $\bar{y}(x)$ of (*) such that $\bar{y}(0) = 0$ is the formal Taylor series at 0 of an infinitely differentiable solution. The proof of Theorem A also gives an equivariant version of Tougeron's theorem.

Theorem A is closely related to the second result of this paper.

THEOREM B. *Suppose that* $G$ *acts linearly on* $V = \mathbf{k}^n$, *and that* $X$ *is a closed algebraic subset of* $V$ *which is invariant under the action of* $G$. *Then there exists a linear action of* $G$ *on a finite dimensional vector space* $Y = \mathbf{k}^q$, *and an equivariant polynomial mapping* $F: V \to Y$ *such that* $X = F^{-1}(0)$.

*If* $\mathbf{k} = \mathbf{R}$ *or* $\mathbf{C}$, *and* $X$ *is a germ at* $0$ *of a closed analytic subset of* $V$ *which is invariant under the action of* $G$, *then there exists a vector space* $Y = \mathbf{k}^q$ *on which* $G$ *acts linearly, and a germ* $F$ *of an equivariant analytic mapping of some neighborhood if* $0 \in V$ *into* $Y$, *such that* $X = F^{-1}(0)$.

A linear action of $G$ on $\mathbf{k}^n$ induces an action on $\mathbf{k}[[x]] = \mathbf{k}[[x_1, \ldots, x_n]]$ (respectively $\mathbf{k}\{x\}$, $\mathbf{k}[x]$) such that

$$(g \cdot f)(x) = f(g^{-1}x)$$

for all $g \in G$ and $f(x) \in \mathbf{k}[[x]]$ (respectively $\mathbf{k}\{x\}$, $\mathbf{k}[x]$). Let $\mathbf{k}[[x]]^G$ (respectively $\mathbf{k}\{x\}^G$, $\mathbf{k}[x]^G$) be the subset of elements fixed by $G$ (the *invariant* elements).

*Remark 1.4.* It is well-known that if $\mathbf{k} = \mathbf{R}$ and $G$ is compact, then the conclusion of Theorem B holds with $F \in (\mathbf{R}[x]^G)^q$ (or $F \in (\mathbf{R}\{x\}^G)^q$ in the analytic case). In general, invariants separate only disjoint Zariski closed invariant subsets of $\mathbf{k}^n$, so that invariant closed algebraic or analytic subsets needn't be defined by invariant equations.

We will prove Theorem B in the following section, considering separately the complex analytic, real analytic, and algebraic cases. These results may also be obtained in a unified way, at least in characteristic zero, from an explicit projection formula related to the Fourier transform (cf. [15], [10, 12.2]). This formula may be of independent interest, and we have included it in section 4. In section 3 we will deduce Theorem A from Theorem B.

The authors enjoyed several conversations with Joseph Becker on the results in this paper.