

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 25 (1979)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS  
**Autor:** Bierstone, Edward / Milman, Pierre  
**Kapitel:** 1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-50374>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# INVARIANT SOLUTIONS OF ANALYTIC EQUATIONS

by Edward BIERSTONE and Pierre MILMAN

## 1. INTRODUCTION

Let  $\mathbf{k}$  be a field of characteristic zero with a non-trivial valuation.

We consider a system of analytic equations

$$(*) \quad f(x, y) = 0,$$

where

$$f(x, y) = (f_1(x, y), \dots, f_q(x, y))$$

are convergent series in the variables

$$x = (x_1, \dots, x_n),$$

$$y = (y_1, \dots, y_p).$$

Suppose that

$$\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_p(x)), \quad \bar{y}_j(x) \in \mathbf{k}[[x]],$$

are formal power series without constant term which solve (\*); i.e. such that  $f(x, \bar{y}(x)) = 0$ . Let  $c$  be a non-negative integer. Artin's approximation theorem [3] asserts that there exists a convergent series solution

$$y(x) = (y_1(x), \dots, y_p(x)), \quad y_j(x) \in \mathbf{k}\{x\},$$

of (\*), such that

$$y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}.$$

Here  $\mathfrak{m}$  denotes the maximal ideal of  $\mathbf{k}[[x]]$ .

Artin also proved an algebraic analogue of this theorem [1]. It says that if  $f(x, y) = 0$  is a system of polynomial equations with formal series solution  $\bar{y}(x)$ , then a series solution  $y(x)$  may be found such that the  $y_j(x)$  are algebraically dependent on  $x_1, \dots, x_n$  (we will say that the  $y_j(x)$  are "algebraic"; cf. [2]). In this analogue  $\mathbf{k}$  is an arbitrary field.

Let  $G$  be a reductive algebraic group (i.e.  $G$  is linear and every rational representation of  $G$  is completely reducible). Suppose that  $G$  acts linearly on  $V = \mathbf{k}^n$  and  $W = \mathbf{k}^p$ . We will say that  $\bar{y}(x) \in \mathbf{k}[[x]]^p$  is *equivariant* if

$$\bar{y}(gx) = g\bar{y}(x), \quad g \in G.$$

We will prove the following theorem.

**THEOREM A.** *Suppose  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , and that  $\bar{y}(x) \in \mathbf{k}[[x]]^p$  is an equivariant formal power series solution of (\*),  $\bar{y}(0) = 0$ . Let  $c \in \mathbf{N}$ . Then there exists an equivariant convergent series solution  $y(x)$  of (\*), such that  $y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}$ .*

*Moreover, if  $f(x, y) = 0$  is a system of polynomial equations (where  $\mathbf{k}$  is any field), then there exists an equivariant algebraic solution  $y(x)$ , such that  $y(x) \equiv \bar{y}(x) \pmod{\mathfrak{m}^c}$ .*

**Remark 1.1.** Theorem A may be regarded in the context of the question: What properties of a formal solution of (\*) may be preserved in an analytic solution? Artin [2] asked whether there is a convergent solution such that some of the variables  $x_i$  are missing in some of the series  $y_j(x)$ , provided there is a formal solution with the same property. Gabrielov [6] answered this question negatively (see also [4]). In [12] it is shown that if a formal solution of a system of real analytic equations satisfies the Cauchy-Riemann equations, then it may be approximated by complex analytic solutions.

**Remark 1.2.** Suppose that  $\pi(x) \in \mathbf{C}\{x\}^r$  is an analytically regular germ of an analytic mapping (terminology of Gabrielov [7]). Let  $F_i(x) \in \mathbf{C}\{x\}^p$ ,  $i = 1, \dots, q$ . We may ask whether formal relations among the  $F_i$  of the form  $(h_1(\pi(x)), \dots, h_q(\pi(x)))$ ; i.e.  $q$ -tuples of formal power series of this form such that

$$\sum_{i=1}^q h_i(\pi(x)) F_i(x) = 0,$$

are generated by analytic relations of the same form. This question generalizes Gabrielov's problem in [7]. The answer is *no* in general, but the method of our proof of Theorem A shows it is *yes* if  $\pi$  is a finite analytic germ. As in our proof of Theorem A, it is then easy to see that a formal solution  $\bar{y}(\pi(x))$  of a system of complex analytic equations  $f(x, y) = 0$  may be approximated by analytic solutions of the same form. We are grateful to Joseph Becker for pointing out the latter result to us.

*Remark 1.3.* Tougeron [16] has proved a generalization of Artin's theorem which asserts, in particular, that every formal solution  $\bar{y}(x)$  of (\*) such that  $\bar{y}(0) = 0$  is the formal Taylor series at 0 of an infinitely differentiable solution. The proof of Theorem A also gives an equivariant version of Tougeron's theorem.

Theorem A is closely related to the second result of this paper.

**THEOREM B.** *Suppose that  $G$  acts linearly on  $V = \mathbf{k}^n$ , and that  $X$  is a closed algebraic subset of  $V$  which is invariant under the action of  $G$ . Then there exists a linear action of  $G$  on a finite dimensional vector space  $Y = \mathbf{k}^q$ , and an equivariant polynomial mapping  $F: V \rightarrow Y$  such that  $X = F^{-1}(0)$ .*

*If  $\mathbf{k} = \mathbf{R}$  or  $\mathbf{C}$ , and  $X$  is a germ at 0 of a closed analytic subset of  $V$  which is invariant under the action of  $G$ , then there exists a vector space  $Y = \mathbf{k}^q$  on which  $G$  acts linearly, and a germ  $F$  of an equivariant analytic mapping of some neighborhood of  $0 \in V$  into  $Y$ , such that  $X = F^{-1}(0)$ .*

A linear action of  $G$  on  $\mathbf{k}^n$  induces an action on  $\mathbf{k}[[x]] = \mathbf{k}[[x_1, \dots, x_n]]$  (respectively  $\mathbf{k}\{x\}$ ,  $\mathbf{k}[x]$ ) such that

$$(g \cdot f)(x) = f(g^{-1}x)$$

for all  $g \in G$  and  $f(x) \in \mathbf{k}[[x]]$  (respectively  $\mathbf{k}\{x\}$ ,  $\mathbf{k}[x]$ ). Let  $\mathbf{k}[[x]]^G$  (respectively  $\mathbf{k}\{x\}^G$ ,  $\mathbf{k}[x]^G$ ) be the subset of elements fixed by  $G$  (the invariant elements).

*Remark 1.4.* It is well-known that if  $\mathbf{k} = \mathbf{R}$  and  $G$  is compact, then the conclusion of Theorem B holds with  $F \in (\mathbf{R}[x]^G)^q$  (or  $F \in (\mathbf{R}\{x\}^G)^q$  in the analytic case). In general, invariants separate only disjoint Zariski closed invariant subsets of  $\mathbf{k}^n$ , so that invariant closed algebraic or analytic subsets needn't be defined by invariant equations.

We will prove Theorem B in the following section, considering separately the complex analytic, real analytic, and algebraic cases. These results may also be obtained in a unified way, at least in characteristic zero, from an explicit projection formula related to the Fourier transform (cf. [15], [10, 12.2]). This formula may be of independent interest, and we have included it in section 4. In section 3 we will deduce Theorem A from Theorem B.

The authors enjoyed several conversations with Joseph Becker on the results in this paper.