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in case of  $n$  odd; the maximal root  $\rho = -\delta_{l-1} - \delta_l$  is real,  $\rho(\overline{S^{-1} H S}) = \rho(H)$ . Hence, the contact structure on  $TG'_0 T^{-1}/TP'_0 T^{-1}$  is that obtained from  $G/P$  by 2.11. We conclude:

$G_0/P_0$  and  $TG'_0 T^{-1}/TP'_0 T^{-1}$ , the latter isomorphic to  $G'_0/P'_0$ , are two real forms of the complex contact manifold  $G/P$ .

5.6. We observed in 5.3 that the space of co-directions in complex projective space  $P^3$ , by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric  $\Omega^4$  in complex  $P^5$ , and that this isomorphism makes real line geometry correspond to a real form of  $\Omega^4$ . We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space  $E^3$  of Lie's higher sphere geometry, which is the space of lines in the quadric  $\Psi^4$  in complex  $P^5$ , is isomorphic to the space of lines in the quadric  $\Omega^4$  also, and that this isomorphism makes real sphere geometry correspond to a second real form of  $\Omega^4$ . That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in  $\Psi^4$  and lines in  $\Omega^4$ . The former places real sphere geometry in the foreground, the latter, real line geometry.

5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex  $P^3$ . Real sphere geometry then leads to the real form  $PSU(2,2)$  of  $PSL(4; \mathbf{C})$ .

#### REFERENCES

- [1] BOOTHBY, W. M. Homogeneous complex contact manifolds. *Proc. Sympos. Pure Math., Vol. III*, pp. 144-154, Amer. Math. Soc., Providence, R. I., 1961.
- [2] — A note on homogeneous complex contact manifolds. *Proc. Amer. Math. Soc. 13* (1961), pp. 276-280.
- [3] BOREL, A. *Linear Algebraic Groups*. W. A. Benjamin, Inc., New York, 1969.
- [4] FREUDENTHAL, H. and H. de VRIES. *Linear Lie Groups*. Academic Press, New York, 1969.
- [5] KLEIN, F. *Lectures on Mathematics, Lectures II and III: Sophus Lie*. The Evanston Colloquium, MacMillan and Co., 1893. Republished by Amer. Math. Soc., New York, 1911.
- [6] — *Vorlesungen über höhere Geometrie*. Springer-Verlag, Berlin 1926. 3. Aufl. reprinted by Chelsea, New York, 1957.

- [7] WOLF, J. A. Complex homogeneous contact manifolds and quaternionic symmetric spaces. *J. Math. Mech.* 14 (1965), pp. 1033-1047.
- [8] — The action of a real semisimple group on a complex flag manifold, I. *Bull. Amer. Math. Soc.* 75 (1969), pp. 1121-1237.
- [9] *Proc. U.S.-Japan Seminar in Differential Geometry*, S. Kôbayashi and J. Eells, Jr. ed., Nippon Hyoronasha, Tokyo, 1966.

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