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5. THE LINE-SPHERE TRANSFORMATION

The homogeneous contact manifold of co-directions in complex projective space P^3 , obtained from the simple complex Lie algebra of type A_3 , must coinside with that of oriented co-directions in complex Euclidean space E^3 , obtained from the algebra of type D_3 , in view of the isomorphisms $A_3 \simeq D_3$. To exhibit this explicitly, we introduce a third homogeneous contact manifold in terms of which both of these can be conveniently described, namely, the space of lines in the quadric Ω^4 in P^5 of Section 1.

5.1. We carry out the construction of 2.10 for the simple complex Lie algebras of type B_l and D_l , making the restriction to type D_3 later.

Let $g = \mathfrak{o}(A; \mathbb{C})$, complex square matrices X for which ${}^{t}XA + AX = 0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_l \\ 0 & 1_l & 0 \end{bmatrix} \text{ in case } B_l$$

or

$$A = \begin{bmatrix} 0 & 1_l \\ 1_l & 0 \end{bmatrix} \text{ in case } D_l$$

that is, the quadratic form defining g is

$${\xi_0}^2 + 2{\xi_1}{\xi_{l+1}} + \dots + 2{\xi_l}{\xi_{2l}}$$

or

$$2\xi_1\xi_{l+1} + \ldots + 2\xi_l\xi_{2l}$$

respectively [4, (16.3) and (16.4)].

We exhibit the details of the construction for the case of D_l . For B_l one need only carry along an additional initial row and column in the matrices, as well as the corresponding roots; the conclusions are the same.

Thus g consists of 21 by 21 matrices of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -tX_1 \end{bmatrix}$$

where X_1 is l by l and arbitrary and X_2 and X_3 are l by l and skew-symmetric. For Cartan subalgebra \mathfrak{h} of \mathfrak{g} take diagonal matrices H of the form

$$H = \operatorname{diag}(h_1, ..., h_l \mid -h_1, ..., -h_l).$$

Let δ_i , i = 1, 2, ..., l be the linear function on \mathfrak{h} which assigns h_i to $H: \delta_i(H) = h_i$. The roots of \mathfrak{g} with respect to \mathfrak{h} are

$$\pm \delta_i \pm \delta_j$$
 $i, j = 1, 2, ..., l$
and $i \neq j$

and the root vector E_{α} corresponding to the root α is

$$E_{\delta_i - \delta_j} = egin{bmatrix} E_{ij} & 0 & & & & \\ 0 & -E_{ji} & & & & \\ E_{\delta_i + \delta_j} & = & & & & \\ 0 & & & & & \\ 0 & & & & & \\ E_{-\delta_i - \delta_j} & = & & & \\ E_{ji} - E_{ij} & & & & \\ \end{bmatrix}, \quad i
eq j,$$

where E_{ij} is the *l* by *l* matrix with 1 in the i^{th} row and j^{th} column and 0s elsewhere [4, (16.3)]. A system of simple roots is

$$\delta_1 - \delta_2$$
, $\delta_2 - \delta_3$, ..., $\delta_{l-1} - \delta_l$, and $-\delta_1 - \delta_2$,

(this is not the same choice as in [4, (16.3)]), for which the maximal root is

$$\rho = -\delta_{l-1} - \delta_{l},$$

[4, App., Table E]. The Killing form of g is $\langle X, Y \rangle = (2l-2)$ tr (XY), but we replace this with $\langle X, Y \rangle = \frac{1}{2}$ tr (XY) for convenience. Then the H_{α} in h are given by

$$H_{\pm\delta_i\pm\delta_j} = \text{diag}(0,...,0,\pm 1,0,...,0,\pm 1,0,...,0)$$

where the $\pm 1s$ occur in the i^{th} and j^{th} entries and the second l entries are the negatives of the first l entries. Especially,

$$H_{\rho} = \text{diag}(0, ..., 0, -1, -1 \mid 0, ..., 0, 1, 1).$$

It is now straightforward to determine for which roots α we have $\langle H_{\rho}, H_{\alpha} \rangle \geqslant 0$ and find that $\mathfrak p$ in (i) of 2.9 consists of matrices of the form

negative transpose of lower right	arbitrary (l-2) by (l-2) skew- symmetric 0	0 0	0 0 0 0	0 0 0	
arbitrary l by l skew- symmetric	*	*	0 **	0	,

where the starred entries are arbitrary.

5.2. The connected centerless simple group $G = PSO(A; \mathbb{C})$ is transitive on the lines of the quadric Ω^{2l-2}

$$\xi_1 \, \xi_{l+1} \, + \, \dots \, + \, \xi_l \, \xi_{2l} \, = \, 0$$

in P^{2l-1} by Witt's theorem. The Lie algebra of the isotropy subgroup of the line l_0 joining

$$t(0, ..., 0, 1, 0)$$
 and $t(0, ..., 0, 0, 1)$

is p. Hence

$$G/P$$
 = space of lines in Ω^{2l-2} .

The element $W = E_{\rho}$ of p giving the contact structure on G/P, as in 2.7, is

In general, the construction of 2.10 gives the (2n-1)-dimensional homogeneous contact manifold of lines in the quadric Ω^{n+1} in P^{n+2} , where Ω^{n+1} is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0$$

in case B_l when n is even, n+3=2l+1, and Ω^{n+1} is Ω^{2l-2} above in case D_l when n is odd, n+3=2l; $n \ge 2$.

The real contact structure on the (2n-1) dimensional space of lines of Ω^{n+1} in real projective space P^{n+2} is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the one- or two- component centerless group $PSO(A; \mathbb{R})$ consisting of real contact automorphisms.

5.3. The line joining $x = {}^t(x_0, x_1, x_2, x_3)$ and $y = {}^t(y_0, y_1, y_2, y_3)$ in complex projective space P^3 has Plücker coordinates $p_{ij} = x_i y_j - x_j y_i$. These coordinates are the coefficients of the bivector $x \wedge y$ with respect to the basis

$$e_1 \wedge e_2$$
, $e_3 \wedge e_1$, $e_2 \wedge e_3$, $e_0 \wedge e_3$, $e_0 \wedge e_2$, $e_0 \wedge e_1$,

where $e_0 = {}^{t}(1, 0, 0, 0), ..., e_3 = {}^{t}(0, 0, 0, 1),$ and satisfy

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0,$$

[6, §69]. If we set

$$\xi_1 = p_{12}, \quad \xi_2 = p_{31}, \quad \xi_3 = p_{23},$$

$$\xi_4 = p_{03}, \quad \xi_5 = p_{02}, \quad \xi_6 = p_{01},$$

we have that the lines of P^3 correspond to the points of the quadric Ω^4

$$\xi_1 \, \xi_4 \, + \, \xi_2 \, \xi_5 \, + \, \xi_3 \, \xi_6 \, = \, 0$$

in P^5 . Two lines of P^3 intersect exactly when their corresponding points on Ω^4 are conjugate, that is, the line joining these points lies entirely in Ω^4 .

To a point x in P^3 we associate all lines of P^3 incident with x and hence a plane lying in Ω^4 . To a plane u in P^3 we associated all lines of P^3 lying in u and hence a plane lying in Ω^4 . These two families of planes doubly rule Ω^4 . To a surface element or co-direction in P^3 , that is, a point x and incident plane u, is then associated all lines of P^3 lying in u and incident with x. In Ω^4 this corresponds to the intersection of the planes corresponding to x and u and is a line. Hence, the 5-dimensional spaces of co-directions in P^3 and lines in Ω^4 correspond naturally.

Note that the co-direction in P^3 consisting of the point $x_0 = {}^t (1, 0, 0, 0)$ and the incident plane $u_0: x_3 = 0$ in 3.2 corresponds to the line l_0 of Ω^4 joining the points ${}^t (0, 0, 0, 0, 1, 0)$ and ${}^t (0, 0, 0, 0, 0, 1)$ in 5.2. For, to the co-direction (x_0, u_0) is associated all lines of P^3 joining x_0 and a point $y = {}^t (y_0, y_1, y_2, 0)$ of u_0 ; such a line has Plücker coordinates

$$\xi_1 = 0$$
, $\xi_2 = 0$, $\xi_3 = 0$,
 $\xi_4 = 0$, $\xi_5 = y_2$, $\xi_6 = y_1$,

and corresponds to a point of Ω^4 lying on l_0 .

The projectivity g in $PSL(4; \mathbb{C})$ permutes the lines of P^3 by $x \wedge y \to gx \wedge gy$, a projectivity of P^5 which preserves Ω^4 . In this way one obtains the isomorphism $A_3 \simeq D_3$:

$$PSL(4; \mathbf{C}) \simeq PSO(A; \mathbf{C}), \quad A = \begin{bmatrix} 0 & 1_3 \\ 1_3 & 0 \end{bmatrix},$$

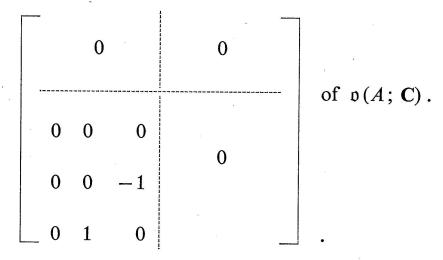
[4, (25.8.4')]. The spaces of co-directions in P^3 and lines in Ω^4 are homogeneous under PSL (4; C) and PSO (A; C) respectively; hence the correspondence between these spaces is as homogeneous spaces. In fact, since (x_0, u_0) and l_0 correspond, their isotropy subgroups, as described in 3.2 and 5.2, correspond under the isomorphism.

From the isomorphism of the groups, we obtain the isomorphism of the Lie algebras $\mathfrak{sl}(4; \mathbb{C}) \simeq \mathfrak{o}(A; \mathbb{C})$, where X in $\mathfrak{sl}(4; \mathbb{C})$ is sent to the linear transformation $x \wedge y \to (Xx) \wedge y + x \wedge (Xy)$ in $\mathfrak{o}(A; \mathbb{C})$. With $X = (a_{ij}), i, j = 0, 1, 2, 3$, the matrix of this transformation with respect to the basis $e_i \wedge e_j$ is

$$\begin{bmatrix} a_{11} + a_{22} & -a_{23} & -a_{13} & 0 & a_{10} & -a_{20} \\ -a_{32} & a_{11} + a_{33} & -a_{12} & -a_{10} & 0 & a_{30} \\ -a_{31} & -a_{21} & a_{22} + a_{33} & a_{20} & -a_{30} & 0 \\ \hline 0 & -a_{01} & a_{02} & a_{00} + a_{33} & a_{32} & a_{31} \\ a_{01} & 0 & -a_{03} & a_{23} & a_{00} + a_{22} & a_{21} \\ -a_{02} & a_{03} & 0 & a_{13} & a_{12} & a_{00} + a_{11} \end{bmatrix};$$

this describes the isomorphism explicitly. Under this isomorphism, the Lie algebras of the isotropy subgroups of (x_0, u_0) and l_0 , as in 3.1 and 5.1, correspond. Moreover, the element

is sent into the element



Since these are the root vectors for the maximal roots which determine the contact structures, as in 3.4 and 5.2, we conclude:

The 5-dimensional manifolds of co-directions in P^3 and lines in Ω^4 are isomorphic as algebraic homogeneous contact manifolds.

This isomorphism holds for the real contact manifolds also; cf. 3.5 and 5.2. The real connected centerless groups $PSL(4; \mathbf{R})$ and $PSO(A; \mathbf{R})$ are isomorphic; each consists of the elements fixed under complex conjugation of matrix entries.

5.4. The algebraic homogeneous contact manifolds of lines in the quadrics Ψ^{n+1} and Ω^{n+1} , 4.3 and 5.2, are isomorphic since they are both obtained from the simple complex Lie algebra of type B_l or D_l by the construction of 2.10. This isomorphism can be exhibited explicitly by means of a contact transformation which reduces to the line-sphere transformation, as described in Section 1, when n=3.

Throughout, unprimed quantities refer to Ω^{n+1} and primed quantities to Ψ^{n+1} . Set n+3=2l+1 or 2l according as n is even or odd; $n \ge 2$.

Thus,

$$G = PSO(A; \mathbf{C}), A = \begin{bmatrix} 1 & 0 & 0 \\ & & & \\ 0 & 0 & 1_{l} \\ & & & \\ 0 & 1_{l} & 0 \end{bmatrix}$$
 or $\begin{bmatrix} 0 & 1_{l} \\ & & \\ & 1_{l} & 0 \end{bmatrix}$

and

$$G' = PSO(A'; \mathbf{C}), A' = \begin{bmatrix} 2 \cdot 1_n & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

These are groups of projectivities preserving Ω^{n+1} and Ψ^{n+1} , respectively, in P^{n+2} .

In case n is odd, the transformation which we consider is

$$\xi_{1} = \alpha_{1} + \sqrt{-1} \alpha_{2} \qquad \qquad \xi_{l+1} = \alpha_{1} - \sqrt{-1} \alpha_{2}$$

$$\xi_{2} = \alpha_{3} + \sqrt{-1} \alpha_{4} \qquad \qquad \xi_{l+2} = \alpha_{3} - \sqrt{-1} \alpha_{4}$$

$$\xi_{l-2} = \alpha_{n-2} + \sqrt{-1} \alpha_{n-1} \qquad \xi_{2l-2} = \alpha_{n-2} - \sqrt{-1} \alpha_{n-1}$$

$$\xi_{l-1} = \alpha_{n} + \lambda \qquad \qquad \xi_{2l-1} = \alpha_{n} - \lambda$$

$$\xi_{1} = \mu \qquad \qquad \xi_{2l} = -\nu.$$

This is a projectivity of P^{n+2} which sends the quadric Ψ^{n+1}

$${\alpha_1}^2 + ... + {\alpha_n}^2 - {\lambda}^2 - \mu v = 0$$

into the quadric Ω^{n+1}

$$2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l} = 0.$$

In case *n* is even, the first equation of the transformation is $\xi_0 = \sqrt{2} \alpha_1$ and the remaining ones are like the above.

As before, we exhibit the details of the calculations for the case of n odd. For n even one need only carry along an additional initial row and column in the matrices; the conclusions are unchanged.

The matrix T of the transformation is

where B is the (l-2) by (2l-4) matrix

and \overline{B} is its complex conjugate; T has inverse

$$T^{-1} = \frac{1}{2} \begin{bmatrix} & {}^{t}\overline{B} & 0 & {}^{t}B & 0 & \\ & & & & & \\ & 1 & 0 & & 1 & 0 \\ & 0 & 1 & 0 & 0 & -1 & 0 \\ & 0 & 2 & & 0 & 0 & \cdot \\ & 0 & 0 & & 0 & -2 & \end{bmatrix}.$$

By direct calculation we ascertain the following:

- (1) $A' = {}^{t}TAT$ and hence $G' = T^{-1}GT$. G and G' are conjugate, but do not coincide, in PSL $(n+3; \mathbb{C})$. As a consequence, $g' = T^{-1}gT$.
- (2) $l'_0 = T^{-1}l_0$; the line l_0 in Ω^{n+1} joining

$$t(0, ..., 0, 1, 0)$$
 and $t(0, ..., 0, 0, 1)$

is sent to the line l_0' of Ψ^{n+1} joining

$$t(0, ..., 0, 0 \mid 0, 0, 1)$$
 and $t(0, ..., 0, 1 \mid -1, 0, 0)$.

Hence their isotropy subgroups, as in 5.2 and 4.3 are conjugate: $P' = T^{-1} P T$. As a consequence, $p' = T^{-1} p T$.

(3) The Cartan subalgebras of g and g' in 5.1 and 4.4 are conjugate: $\mathfrak{h}' = T^{-1} \mathfrak{h} T$. In fact, for

$$H = \operatorname{diag}(h_1, ..., h_l \mid -h_1, ..., -h_l)$$

in h, we have

$$T^{-1}HT = \operatorname{diag} \begin{bmatrix} 0 & \sqrt{-1}h_1 \\ -\sqrt{-1}h_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \sqrt{-1}h_{l-2} \\ -\sqrt{-1}h_{l-2} & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix}$$

in h'.

4) The elements W and W' of the Lie algebras which give the contact structures on G/P and G'/P', as in 5.2 and 4.4, are conjugate: $W' = T^{-1} W T$. We conclude:

The (2n-1)-dimensional manifolds of lines in Ω^{n+1} and lines in Ψ^{n+1} are isomorphic as algebraic homogeneous contact manifolds. The isomorphism is a consequence of the projectivity T carrying Ψ^{n+1} into Ω^{n+1} . T sends lines of Ψ^{n+1} into lines of Ω^{n+1} and is a contact transformation.

5.5. $G_0 = PSO(A; \mathbf{R})$ is a real form of G; it consists of the elements of G fixed under the conjugation $g \to \overline{g}$ of G, complex conjugation of the matrix entries of g. The Cartan subalgebra \mathfrak{h} of \mathfrak{g} , as in 5.1, is stable and the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real. With $P_0 = G_0 \cap P$, we obtain from 2.11 the real contact manifold

 G_0/P_0 = space of lines in Ω^{n+1} in real P^{n+2} ,

a real form of G/P; cf. 5.2. The same remarks apply to the real form $G_0' = PSO(A'; \mathbf{R})$ of G' for the conjugation $g' \to \bar{g}'$. With $P_0' = G_0' \cap P'$, we obtain the real contact manifold

 G'_0/P'_0 = space of lines in Ψ^{n+1} in real P^{n+2}

- = space of pencils of mutually tangent oriented spheres in real E^n
- = space of oriented co-directions in real E^n ,

a real form of G/P; cf. 4.7.

Since $G' = T^{-1} G T$, we can exhibit G'_0/P'_0 , as well as G_0/P_0 , as a real form of the complex contact manifold G/P. $T G'_0 P^{-1}$ is the real form of $G = T G' T^{-1}$ consisting of the elements fixed under the conjugation obtained by transporting the conjugation $g' \to \bar{g}'$ of G' to G, namely

$$g \to T \overline{(T^{-1}gT)} T^{-1} = S^{-1}\bar{g}S$$

where $S = \overline{T}T^{-1}$. In the case of n odd,

$$S = \begin{bmatrix} 0 & 0 & 1_{l-2} & 0 \\ 0 & 1_2 & 0 & 0 \\ & & & & \\ 1_{l-2} & 0 & 0 & 0 \\ & 0 & 0 & 1_2 \end{bmatrix};$$

in the case of n even, S has an additional initial row and column with a 1 in their common first entry and 0s elsewhere. ${}^tSAS = A$ and $S^2 = 1_{n+3}$, so the complex conjugation $\xi \to S^{-1}\overline{\xi}$ preserves the quadric Ω^{n+1} . A point or line of Ω^{n+1} is fixed under this conjugation exactly if it is the image under T of a real point or line of Ψ^{n+1} . The latter constitute the orbit on Ω^{n+1} of $TG_0'T^{-1}$. The isotropy subgroup in $TG_0'T^{-1}$ of the line l_0 of Ω^{n+1} is $TG_0'T^{-1} \cap P = TP_0'T^{-1}$. Furthermore, the Cartan subalgebra Ω of Ω in 5.1 is stable under the conjugation $X \to S^{-1} \overline{X} S$ of Ω ; in fact, for

$$H = \text{diag}(h_1, ..., h_l \mid -h_1, ..., -h_l)$$

in h, we have

$$S^{-1}\overline{H} S = \text{diag} (-\overline{h}_1, ..., -\overline{h}_{l-2}, \overline{h}_{l-1}, \overline{h}_l \mid \overline{h}_1, ..., \overline{h}_{l-2}, -\overline{h}_{l-1}, -\overline{h}_l),$$

in case of n odd; the maximal root $\rho = -\delta_{l-1} - \delta_l$ is real, $\rho(S^{-1} \overline{H} S) = \rho(H)$. Hence, the contact structure on $TG_0'T^{-1}/TP_0'T^{-1}$ is that obtained from G/P by 2.11. We conclude:

 G_0/P_0 and $TG_0'T^{-1}/TP_0'T^{-1}$, the latter isomorphic to G_0'/P_0' , are two reals forms of the complex contact manifold G/P.

- 5.6. We observed in 5.3 that the space of co-directions in complex projective space P^3 , by means of Plücker's line geometry, is isomorphic to the space of lines in the quadric Ω^4 in complex P^5 , and that this isomorphism makes real line geometry correspond to a real form of Ω^4 . We found in 5.4 and 5.5 that the space of oriented co-directions in complex Euclidean space E^3 of Lie's higher sphere geometry, which is the space of lines in the quadric Ψ^4 in complex P^5 , is isomorphic to the space of lines in the quadric Ω^4 also, and that this isomorphism makes real sphere geometry correspond to a second real form of Ω^4 . That is, real line geometry and real sphere geometry are two distinct real forms of complex line geometry. The line-sphere transformation establishes the isomorphism of the spaces of lines in Ψ^4 and lines in Ω^4 . The former places real sphere geometry in the foreground, the latter, real line geometry.
- 5.7. The isomorphism of 5.3 may be used to describe sphere geometry in terms of co-directions in complex P^3 . Real sphere geometry then leads to the real form PSU(2,2) of $PSL(4; \mathbb{C})$.

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