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and the $n0$ -entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1}.$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the $(2n-1)$ -dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. HIGHER SPHERE GEOMETRY

4.1 In complex Euclidean space E^n , the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center (a_1, \dots, a_n) and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C.$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the $n+2$ coordinates a_1, \dots, a_n, r, C , which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0,$$

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{v}, \quad r = \frac{\lambda}{v}, \quad C = \frac{\mu}{v},$$

$i = 1, 2, \dots, n$. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu v = 0.$$

The sphere corresponding to the point $(\alpha_1, \dots, \alpha_n, \lambda, \mu, v)$ of Ψ^{n+1} is

$$v(x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0.$$

Ordinary spheres have finite nonzero radius r , so $v \neq 0$. For $v = 0$, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no

orientation. If we include these special cases as spheres of E^n , then Ψ^{n+1} is the space of all oriented spheres in E^n .

Two spheres in E^n with centers (a_1, \dots, a_n) , (a'_1, \dots, a'_n) and radii r, r' respectively are tangent, orientations taken into account, if

$$(a_1 - a'_1)^2 + \dots + (a_n - a'_n)^2 = (r - r')^2.$$

Use $a_1^2 + \dots + a_n^2 = r^2 + C$ for both spheres to obtain the condition for tangency as

$$2a_1 a'_1 + \dots + 2a_n a'_n - 2rr' - C - C' = 0$$

or, in homogeneous coordinates,

$$2\alpha_1 \alpha'_1 + \dots + 2\alpha_n \alpha'_n - 2\lambda \lambda' - \mu \nu' - \nu \mu' = 0.$$

Hence, two spheres of E^n are tangent when their corresponding points in Ψ^{n+1} are conjugate, that is, the line joining these points lies entirely in Ψ^{n+1} [6, §25].

4.2. A pencil of mutually tangent spheres in E^n corresponds to a line in Ψ^{n+1} . This pencil of spheres determines an "oriented complex co-direction" in E^n since it contains a point sphere and an incident oriented hyperplane. Corresponding to the hyperplane

$$x'_n - x_n = p_1 (x'_1 - x_1) + \dots + p_{n-1} (x'_{n-1} - x_{n-1})$$

at the point (x_1, \dots, x_n) is the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ \hline -\sqrt{pp+1} \\ 2(x_n - px) \\ 0 \end{bmatrix}$$

of Ψ^{n+1} , where

$$xx = \sum_{i=1}^n x_i^2, \quad px = \sum_{i=1}^{n-1} p_i x_i, \quad pp = \sum_{i=1}^{n-1} p_i^2;$$

this is the pencil of spheres

$$\sum_{i=1}^{n-1} (x'_i - x_i + t p_i)^2 + (x'_n - x_n - t)^2 = t^2 \left(\sum_{i=1}^{n-1} p_i^2 + 1 \right)$$

passing through (x_1, \dots, x_n) and having their centers on the line normal to the hyperplane at this point.

For later calculations it will be convenient to replace $-p, \dots, -p_{n-1}, 1$ by homogeneous u_1, \dots, u_{n-1}, u_n . The line in Ψ^{n+1} corresponding to the hyperplane

$$u_1(x'_1 - x_1) + \dots + u_n(x'_n - x_n) = 0$$

at the point (x_1, \dots, x_n) is then

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix},$$

where

$$xx = \sum_{i=1}^n x_i^2, \quad ux = \sum_{i=1}^n u_i x_i, \quad uu = \sum_{i=1}^n u_i^2.$$

Any convenient condition may be imposed on uu .

4.3. The contact structure on the $(2n-1)$ -dimensional space of lines in Ψ^{n+1} , that is, the space of oriented co-directions in complex Euclidean space E^n , is obtained when the construction of 2.10. is carried out for the simple complex Lie algebra of type B_l or D_l , $l \geq 2$ and $l \geq 3$ respectively. However, it will be simpler to identify quantities geometrically if

we proceed by using the description of 2.7, since now the groups are determined first.

Let

$$A = \left[\begin{array}{c|ccc} & & & & \\ & 2 \cdot 1_n & & & 0 \\ \hline & & & & \\ & & & -2 & 0 & 0 \\ & & 0 & 0 & 0 & -1 \\ & & & 0 & -1 & 0 \end{array} \right]$$

be the matrix of the quadratic form defining Ψ^{n+1} in P^{n+1} . $SO(A; \mathbb{C})$, the special orthogonal group of this form, consists of matrices g in $SL(n+3; \mathbb{C})$ for which ${}^t g A g = A$. The connected centerless simple group $G = PSO(A; \mathbb{C}) = SO(A; \mathbb{C})/\{\text{center}\}$ is transitive on the lines of Ψ^{n+1} by Witt's theorem. Let l_0 be the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline -1 \\ 0 \\ 0 \end{bmatrix}$$

of Ψ^{n+1} , joining

$${}^t(0, \dots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \dots, 0, 1 \mid -1, 0, 0);$$

this corresponds to the pencil of spheres

$$\sum_{i=1}^{n-1} x_i'^2 + (x_n' - t)^2 = t^2$$

tangent to the hyperplane $x_n = 0$ at the origin of E^n , suitably oriented, as in 4.2. Let P denote the isotropy subgroup of l_0 . Then

G/P = space of lines in Ψ^{n+1}

= space of pencils of mutually tangent oriented spheres in E^n

= space of oriented co-directions in complex E^n .

The Lie algebra \mathfrak{g} of G consists of $(n+3)$ by $(n+3)$ matrices X for which ${}^tXA + AX = 0$. The matrices of \mathfrak{g} are of the form

$$\left[\begin{array}{ccc|ccc} & & & b_1 & c_1 & d_1 \\ & & & \vdots & \vdots & \vdots \\ n \text{ by } n \text{ skew-} & & & b_{n-1} & c_{n-1} & d_{n-1} \\ \text{symmetric} & & & b_n & c_n & d_n \\ \hline b_1 \dots b_{n-1} & b_n & 0 & c & d \\ 2d_1 \dots 2d_{n-1} & 2d_n & -2d & e & 0 \\ 2c_1 \dots 2c_{n-1} & 2c_n & -2c & 0 & -e \end{array} \right].$$

P consists of those elements of G which send the subspace of \mathbb{C}^{n+3} spanned by

$${}^t(0, \dots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \dots, 0, 1 \mid -1, 0, 0)$$

into itself; the Lie algebra \mathfrak{p} of P consists of those elements of \mathfrak{g} which do the same. Hence, the matrices of \mathfrak{p} are of the form

$$\left[\begin{array}{ccc|ccc} & & & b_1 & b_1 & c_1 & 0 \\ & & & \vdots & \vdots & \vdots & \vdots \\ (n-1) \text{ by } (n-1) & & & b_{n-1} & b_{n-1} & c_{n-1} & 0 \\ \text{skew-symmetric} & & & & & & \\ \hline -b_1 \dots -b_{n-1} & 0 & b_n & c_n & -d \\ b_1 \dots b_{n-1} & b_n & 0 & c & d \\ 0 \dots 0 & -2d & -2d & e & 0 \\ 2c_1 \dots 2c_{n-1} & 2c_n & -2c & 0 & -e \end{array} \right].$$

Note that \mathfrak{g} and \mathfrak{p} have dimensions $\frac{1}{2}(n+3)(n+2)$ and $\frac{1}{2}(n-1)(n-2) + 2n + 3 = \frac{1}{2}(n+3)(n+2) - 2n + 1$, respectively, in agreement with G/P having dimension $2n-1$.

4.4. For $n \geq 2$, set $n+3 = 2l+1$ or $2l$ according as n is even or odd. \mathfrak{g} is of type B_l or D_l , $l \geq 2$ and $l \geq 3$ respectively.

For Cartan subalgebra \mathfrak{h} of \mathfrak{g} take matrices of the form

$$H = \text{diag} \left[0, \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & h_{l-2} \\ -h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix} \right];$$

the first row and column occur only in case B_l , it is suppressed for case D_l . The Killing form of \mathfrak{g} is $\langle X, Y \rangle = (n+1) \text{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY)$ for convenience.

Let W in \mathfrak{p} be

$$W = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 - \frac{1}{2} 0 \\ 0 & 0 & 0 \frac{1}{2} 0 \\ 0 & 0 & 0 0 0 \\ -1 & -1 & 0 0 \end{bmatrix}$$

For H in \mathfrak{h} we have $[H, W] = -(h_{l-1} + h_l) W$, so $\rho(H) = -(h_{l-1} + h_l)$ is a root of \mathfrak{g} with respect to \mathfrak{h} and $W = E_\rho$ is the corresponding root vector.

For X in \mathfrak{g} as described in 4.3, direct calculation shows $[X, W] = 0$ implies X is in \mathfrak{p} and $b_n + e = 0$; thus the centralizer of W in \mathfrak{g} consists of those elements of \mathfrak{p} with $b_n + e = 0$. For X in \mathfrak{p} now, the same calculation gives $[X, W] = -(b_n + e) W$, so $[X, W] = \rho(X) W$ with ρ extended to \mathfrak{p} by $\rho(X) = -(b_n + e)$. Finally, W is orthogonal to \mathfrak{p} with respect to the Killing form. Hence, (a', c', b') of 2.7 are satisfied, and W is the element of \mathfrak{g} giving the contact structure on G/P .

The origin of the element W is not immediately evident. It was obtained by determining the maximal root and corresponding root vector for Lie algebras of type B_l and D_l when the quadratic form is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

and then passing to the form

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu\nu$$

by conjugating by the element of $PSL(n+3; \mathbb{C})$ which corresponds to the "line-sphere transformation". This will be described further in the next section.

4.5. Let \mathfrak{m} be the $(2n-1)$ -dimensional supplement to \mathfrak{p} in \mathfrak{g} consisting of matrices of the form

$$\left[\begin{array}{cccc|ccc} & & & -b_1 & b_1 & 0 & d_1 \\ & & & \vdots & \vdots & \vdots & \vdots \\ & 0 & & -b_{n-1} & b_{n-1} & 0 & d_{n-1} \\ & & & \vdots & \vdots & \vdots & \vdots \\ b_1 & \dots & b_{n-1} & 0 & 0 & 0 & d_n \\ \hline b_1 & \dots & b_{n-1} & 0 & 0 & 0 & d_n \\ 2d_1 & \dots & 2d_{n-1} & 2d_n & -2d_n & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

cf. 2.12. For X in \mathfrak{m} we have

$$X^2 = \begin{bmatrix} 0 & & & & 0 \\ & -bb & & & bb \ 0 \ bd \\ & & & & \\ \hline & -bb & & & bb \ 0 \ bd \\ 0 & -2bd & & 2bd \ 0 \ 2dd \\ & 0 & & 0 \ 0 \ 0 \end{bmatrix}$$

where

$$bb = \sum_{i=1}^{n-1} b_i^2, \quad bd = \sum_{i=1}^{n-1} b_i d_i, \quad dd = \sum_{i=1}^{n-1} d_i^2.$$

The product of any three matrices of \mathfrak{m} is zero. Especially,

$$\exp X = 1_{n+3} + X + \frac{1}{2} X^2.$$

In order to establish classically identifiable coordinates on G/P as in 2.12, we must determine X in \mathfrak{m} so that $(\exp X) \cdot l_0$ is the line of Ψ^{n+1} described in 4.2. With X in \mathfrak{m} as above, $(\exp X) \cdot l_0$ is the line joining the points

$$(\exp X) \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n + \frac{1}{2} bd \\ \hline d_n + \frac{1}{2} bd \\ dd \\ 1 \end{bmatrix}$$

and

$$(\exp X) \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1 - bb \\ \hline -1 - bb \\ 4d_n - 2bd \\ 0 \end{bmatrix}.$$

On this line we can identify the point sphere when $\lambda = 0$, giving

$$\begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n + \frac{1}{2} bd \\ \hline d_n + \frac{1}{2} bd \\ dd \\ 1 \end{bmatrix} + \frac{d_n + \frac{1}{2} bd}{1 + bb} \begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1 - bb \\ \hline -1 - bb \\ 4d_n - 2bd \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \\ 1 \end{bmatrix},$$

and the incident oriented hyperplane when $v = 0$, giving

$$\begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hline -1-bb \\ 4d_n-2bd \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix}.$$

These equations will be satisfied if we impose the condition $\sqrt{uu} = 1 + bb$ on uu , or

$$u_i = -2b_i, \quad u_n = 1 - bb,$$

$i = 1, 2, \dots, n-1$, and then set

$$b_i = -\frac{1}{2} u_i,$$

$$d_i = x_i - \frac{1}{2} u_i x_n, \quad i = 1, 2, \dots, n-1$$

$$d_n = \frac{1}{4} \sum_{i=1}^{n-1} u_i x_i + \frac{1}{2} x_n.$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, \dots, x_n, u_1, \dots, u_n$ on G/P as in 2.12 and 4.2.

4.6. From 2.12, the form ω on G/P is obtained as

$$\omega = \langle W, (\exp X)^{-1} d(\exp X) \rangle$$

with $(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2} [X, dX].$

Take X as in 4.5 and let the entries of dX be denoted as those of X with primes affixed. Then

$$(\exp X)^{-1} d (\exp X) = \begin{bmatrix} & & -b_1' & b_1' & 0 & d_1' \\ & 0 & \vdots & \vdots & \vdots & \vdots \\ & & -b_{n-1}' & b_{n-1}' & 0 & d_{n-1}' \\ b_1' \dots b_{n-1}' & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\ \hline b_1' \dots b_{n-1}' & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\ 2d_1' \dots 2d_{n-1}' & 2d_n' - c & -2d_n' + c & 0 & 0 \\ 0 \dots 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$c = \sum_{i=1}^{n-1} (b_i d_i' - d_i b_i'),$$

and consequently, from the definition of W in 4.4, $\omega = c - 2d_n'$. Using the expressions in 4.5 for $b_1, \dots, b_{n-1}, d_1, \dots, d_n$ in terms of $x_1, \dots, x_n, u_1, \dots, u_n$, we obtain

$$\omega = - \sum_{i=1}^{n-1} u_i dx_i - \left[1 - \frac{1}{4} \sum_{i=1}^{n-1} u_i^2 \right] dx_n$$

or, since $1 - \frac{1}{4} \sum_{i=1}^{n-1} u_i^2 = u_n$,

$$\omega = - (u_1 dx_1 + \dots + u_n dx_n).$$

This identifies the contact structure with the classical one as in 2.12 and 4.2.

4.7. The real contact structure on the $(2n-1)$ -dimensional space of oriented co-direction in real Euclidean space E^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the two-component centerless group $PSO(A; \mathbf{R})$ consisting of real contact automorphisms.