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and the *n*0-entry is

$$\omega = dx_n - p_1 dx_1 - \ldots - p_{n-1} dx_{n-1}.$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the (2n-1)-dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. HIGHER SPHERE GEOMETRY

4.1 In complex Euclidean space E^n , the equation

$$x_{1}^{\prime 2} + \dots + x_{n}^{\prime 2} - 2a_{1}x_{1}^{\prime} - \dots - 2a_{n}x_{n}^{\prime} + C = 0$$

describes a sphere with center $(a_1, ..., a_n)$ and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C \, .$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the n+2 coordinates $a_1, ..., a_n, r, C$, which are related by

$$a_1^2 + \ldots + a_n^2 - r^2 - C = 0$$
,

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{v}, \ r = \frac{\lambda}{v}, \ C = \frac{\mu}{v},$$

i = 1, 2, ..., n. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \ldots + \alpha_n^2 - \lambda^2 - \mu v = 0.$$

The sphere corresponding to the point $(\alpha_1, ..., \alpha_n, \lambda, \mu, \nu)$ of Ψ^{n+1} is

$$y(x_1^{\prime 2} + ... + x_n^{\prime 2}) - 2\alpha_1 x_1^{\prime} - ... - 2\alpha_n x_n^{\prime} + \mu = 0.$$

Ordinary spheres have finite nonzero radius r, so $v \neq 0$. For v = 0, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no

orientation. If we include these special cases as spheres of E^n , then Ψ^{n+1} is the space of all oriented spheres in E^n .

Two spheres in E^n with centers $(a_1, ..., a_n)$, $(a'_1, ..., a'_n)$ and radii r, r' respectively are tangent, orientations taken into account, if

$$(a_1 - a'_1)^2 + \dots + (a_n - a'_n)^2 = (r - r')^2$$
.

Use $a_1^2 + ... + a_n^2 = r^2 + C$ for both spheres to obtain the condition for tangency as

 $2a_1a'_1 + \dots + 2a_na'_n - 2rr' - C - C' = 0$

or, in homogeneous coordinates,

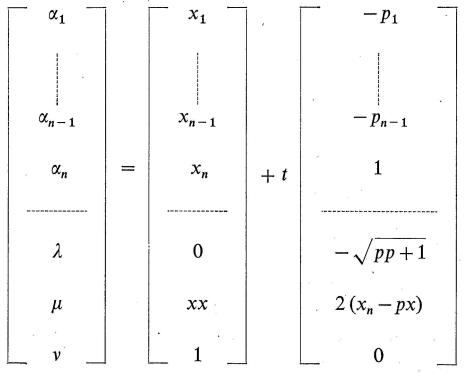
$$2\alpha_1\alpha_1' + \ldots + 2\alpha_n\alpha_n' - 2\lambda\lambda' - \mu\nu' - \nu\mu' = 0$$

Hence, two spheres of E^n are tangent when their corresponding points in Ψ^{n+1} are conjugate, that is, the line joining these points lies entirely in Ψ^{n+1} [6, §25].

4.2. A pencil of mutually tangent spheres in E^n corresponds to a line in Ψ^{n+1} . This pencil of spheres determines an "oriented complex co-direction" in E^n since it contains a point sphere and an incident oriented hyperplane. Corresponding to the hyperplane

$$x'_{n} - x_{n} = p_{1}(x'_{1} - x_{1}) + \dots + p_{n-1}(x_{n-1} - x_{n-1})$$

at the point $(x_1, ..., x_n)$ is the line



of Ψ^{n+1} , where

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$$xx = \sum_{i=1}^{n} x_i^2$$
, $px = \sum_{i=1}^{n-1} p_i x_i$, $pp = \sum_{i=1}^{n-1} p_i^2$

this is the pencil of spheres

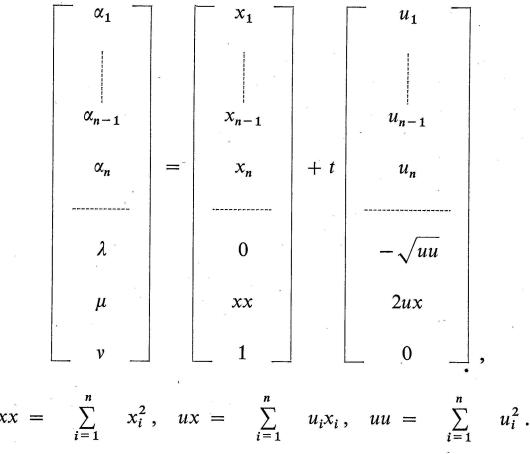
$$\sum_{i=1}^{n-1} (x'_i - x_i + tp_i)^2 + (x'_n - x_n - t)^2 = t^2 \left(\sum_{i=1}^{n-1} p_i^2 + 1\right)$$

passing through $(x_1, ..., x_n)$ and having their centers on the line normal to the hyperplane at this point.

For later calculations it will be convenient to replace $-p, ..., -p_{n-1}, 1$ by homogeneous $u_1, ..., u_{n-1}, u_n$. The line in Ψ^{n+1} corresponding to the hyperplane

$$u_1(x'_1-x_1) + \dots + u_n(x'_n-x_n) = 0$$

at the point $(x_1, ..., x_n)$ is then



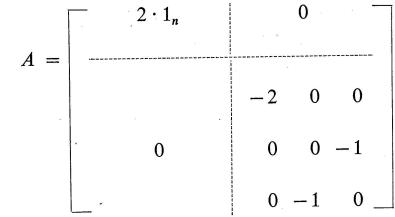
where

 $xx = \sum_{i=1}^{n} x_i^2, \quad ux = \sum_{i=1}^{n} u_i x_i, \quad uu = \sum_{i=1}^{n} u_i x_i$

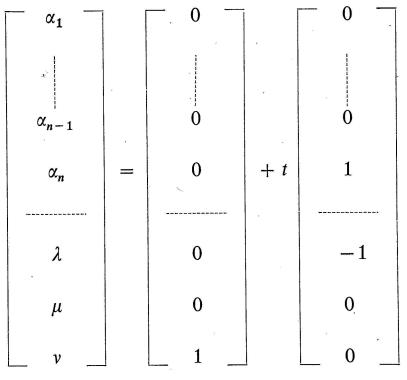
Any convenient condition may be imposed on uu.

The contact structure on the (2n-1)-dimensional space of lines 4.3. in Ψ^{n+1} , that is, the space of oriented co-directions in complex Euclidean space E^n , is obtained when the construction of 2.10. is carried out for the simple complex Lie algebra of type B_l or D_l , $l \ge 2$ and $l \ge 3$ respectively. However, it will be simpler to identify quantities geometrically if

we proceed by using the description of 2.7, since now the groups are determined first.



be the matrix of the quadratic form defining Ψ^{n+1} in P^{n+1} . SO $(A; \mathbb{C})$, the special orthogonal group of this form, consists of matrices g in $SL(n+3; \mathbb{C})$ for which ${}^{t}gAg = A$. The connected centerless simple group $G = PSO(A; \mathbb{C}) = SO(A; \mathbb{C})/\{\text{center}\}$ is transitive on the lines of Ψ^{n+1} by Witt's theorem. Let l_0 be the line



of Ψ^{n+1} , joining

Let

 ${}^{t}(0,...,0,0 \mid 0,0,1)$ and ${}^{t}(0,...,0,1 \mid -1,0,0);$

this corresponds to the pencil of spheres

$$\sum_{i=1}^{n-1} x_i^{\prime 2} + (x_n^{\prime} - t)^2 = t^2$$

tangent to the hyperplane $x_n = 0$ at the origin of E^n , suitably oriented, as in 4.2. Let P denote the isotropy subgroup of l_0 . Then

G/P = space of lines in Ψ^{n+1}

= space of pencils of mutually tangent oriented spheres in E^n

= space of oriented co-directions in complex E^n .

The Lie algebra g of G consists of (n+3) by (n+3) matrices X for which ${}^{t}XA + AX = 0$. The matrices of g are of the form

n by n skew- symmetric	b_1 b_{n-1} b_n			
$b_1 \dots b_{n-1} b_n$	0	С	d	
$2d_1 \dots 2d_{n-1} \ 2d_n$	-2d	е	. 0	
$\underline{} 2c_1 \ldots 2c_{n-1} 2c_n$	-2c	0	<u>-</u> e	

P consists of those elements of *G* which send the subspace of \mathbb{C}^{n+3} spanned by ${}^{t}(0, ..., 0, 0 \mid 0, 0, 1)$ and ${}^{t}(0, ..., 0, 1 \mid -1, 0, 0)$

into itself; the Lie algebra p of P consists of those elements of g which do the same. Hence, the matrices of p are of the form

Note that g and p have dimensions $\frac{1}{2}(n+3)(n+2)$ and $\frac{1}{2}(n-1)(n-2)$

+ $2n + 3 = \frac{1}{2}(n+3)(n+2) - 2n + 1$, respectively, in agreement with G/P having dimension 2n-1.

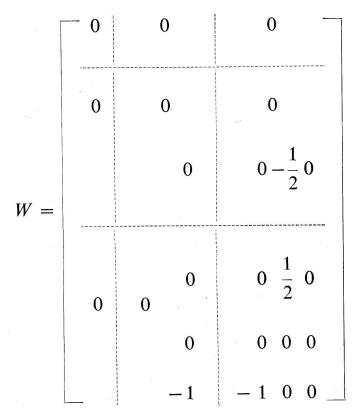
4.4. For $n \ge 2$, set n + 3 = 2l + 1 or 2l according as n is even or odd. g is of type B_l or D_l , $l \ge 2$ and $l \ge 3$ respectively.

For Cartan subalgebra h of g take matrices of the form

$$H = \text{diag} \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & h_{l-2} \\ -h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix};$$

the first row and column occur only in case B_l , it is suppressed for case D_l . The Killing form of g is $\langle X, Y \rangle = (n+1)$ tr (XY), but we replace this with $\langle X, Y \rangle = \frac{1}{2}$ tr (XY) for convenience.

Let W in p be



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For H in h we have $[H, W] = -(h_{l-1}+h_l) W$, so $\rho(H) = -(h_{l-1}+h_l)$ is a root of g with respect to h and $W = E_{\rho}$ is the corresponding root vector.

For X in g as described in 4.3, direct calculation shows [X, W] = 0implies X is in p and $b_n + e = 0$; thus the centralizer of W in g consists of those elements of p with $b_n + e = 0$. For X in p now, the same calculation gives $[X, W] = -(b_n + e) W$, so $[X, W] = \rho(X) W$ with ρ extended to p by $\rho(X) = -(b_n + e)$. Finally, W is orthogonal to p with respect to the Killing form. Hence, (a', c', b') of 2.7 are satisfied, and W is the element of g giving the contact structure on G/P.

The origin of the element W is not immediately evident. It was obtained by determining the maximal root and corresponding root vector for Lie algebras of type B_l and D_l when the quadratic form is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

and then passing to the form

$$\alpha_1^2 + \ldots + \alpha_n^2 - \lambda^2 - \mu v$$

by conjugating by the element of $PSL(n+3; \mathbb{C})$ which corresponds to the "line-sphere transformation". This will be described further in the next section.

4.5. Let m be the (2n-1)-dimensional supplement to p in g consisting of matrices of the form

· ·	$-b_1$	$b_1 0 d_1$	
	$-b_{n-1}$	$b_{n-1} 0 d_{n-1}$	
$b_1 \dots b_{n-1}$	0	$0 0 d_n$	
$b_1 \ \dots \ b_{n-1}$	0	$0 0 d_n$	
$2d_1 \dots 2d_{n-1}$	$2d_n$	$-2d_n 0 0$	
0 0	0	0 0 0 _	,

cf. 2.12. For X in m we have

$$X^{2} = \begin{bmatrix} 0 & 0 & 0 \\ -bb & bb & 0 & bd \\ & -bb & bb & 0 & bd \\ 0 & -2bd & 2bd & 0 & 2dd \\ & 0 & 0 & 0 & -2bd & 0 & 0 & 0 \end{bmatrix}$$

where

$$bb = \sum_{i=1}^{n-1} b_i^2, \ bd = \sum_{i=1}^{n-1} b_i d_i, \ dd = \sum_{i=1}^{n-1} d_i^2$$

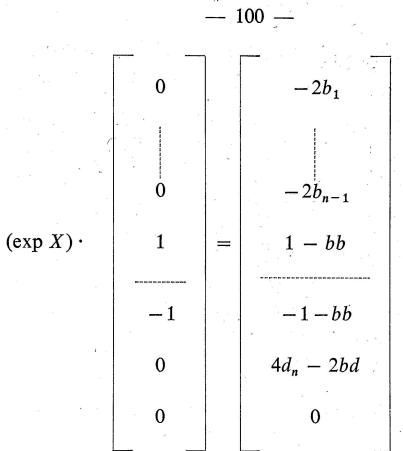
The product of any three matrices of m is zero. Especially,

$$\exp X = 1_{n+3} + X + \frac{1}{2} X^2.$$

In order to establish classically identifiable coordinates on G/P as in 2.12, we must determine X in m so that $(\exp X) \cdot l_0$ is the line of Ψ^{n+1} described in 4.2. With X in m as above, $(\exp X) \cdot l_0$ is the line joining the points

$$(\exp X) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_{n-1} \\ d_n + \frac{1}{2} bd \\ d_n + \frac{1}{2} bd \\ dd \\ dd \\ 1 \end{bmatrix}$$

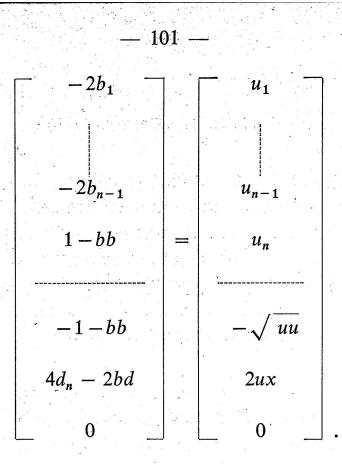
and



On this line we can identify the point sphere when $\lambda = 0$, giving

 d_1 $-2b_{1}$ x_1 $-2b_{n-1}$ d_{n-1} x_{n-1} $+ \frac{d_n + \frac{1}{2}bd}{1+bb}$ $d_n + \frac{1}{2}bd$ 1 - bb x_n = $d_n+\frac{1}{2}bd$ -1 - bb0 $4d_n - 2bd$ dd xx . 1 0 1

and the incident oriented hyperplane when v = 0, giving



These equations will be satisfied if be we impose the condition $\sqrt{uu} = 1 + bb$ on *uu*, or

$$u_i = -2b_i, \quad u_n = 1 - bb$$
,

$$i = 1, 2, ..., n-1$$
, and then set

$$b_{i} = -\frac{1}{2}u_{i},$$

$$d_{i} = x_{i} - \frac{1}{2}u_{i}x_{n},$$

$$i = 1, 2, ..., n-1$$

$$d_{n} = \frac{1}{4}\sum_{i=1}^{n-1}u_{i}x_{i} + \frac{1}{2}x_{n}.$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, ..., x_n, u_1, ..., u_n$ on G/P as in 2.12 and 4.2.

4.6. From 2.12, the form ω on G/P is obtained as $\omega = \langle W, (\exp X)^{-1} d(\exp X) \rangle$

with $(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2} [X, dX].$

Take X as in 4.5 and let the entries of dX be denoted as those of X with primes affixed. Then

$$-102 - \frac{b_{1}'}{b_{1}'} + \frac{b_{1}'}{0} + \frac{b_{1}'}{b_{1}'} + \frac{$$

where

 $(\exp X)^{-1} d (\exp X) =$

$$c = \sum_{i=1}^{n-1} (b_i d_i' - d_i b_i'),$$

and consequently, from the definition of W in 4.4, $\omega = c - 2d'_n$. Using the expressions in 4.5 for $b_1, ..., b_{n-1}, d_1, ..., d_n$ in terms of $x_1, ..., x_n, u_1, ..., u_n$, we obtain

$$\omega = -\sum_{i=1}^{n-1} u_i dx_i - \left[1 - \frac{1}{4} \sum_{i=1}^{n-1} u_i^2\right] dx_n$$

or, since $1 - \frac{1}{4} \sum_{i=1}^{n-1} u_i^2 = u_n$,

 $\omega = -(u_1 \, dx_1 + \dots + u_n \, dx_n) \, .$

This identifies the contact structure with the classical one as in 2.12 and 4.2.

4.7. The real contact structure on the (2n-1)-dimensional space of oriented co-direction in real Euclidean space E^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the two-component centerless group PSO (A; **R**) consisting of real contact automorphisms.