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and the $n0$-entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1} \, .$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the $(2n-1)$-dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. Higher sphere geometry

4.1 In complex Euclidean space E^n, the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center $(a_1, \dots, a_n)$ and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C \, .$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the $n+2$ coordinates $a_1, \dots, a_n, r, C$, which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0 \, ,$$

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{\nu} \, , \quad r = \frac{\lambda}{\nu} \, , \quad C = \frac{\mu}{\nu} \, ,$$

$i = 1, 2, \dots, n$. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu\nu = 0 \, .$$

The sphere corresponding to the point $(\alpha_1, \dots, \alpha_n, \lambda, \mu, \nu)$ of Ψ^{n+1} is

$$\nu (x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0 \, .$$

Ordinary spheres have finite nonzero radius r, so $\nu \neq 0$. For $\nu = 0$, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no orientation. If we include these special cases as spheres of E^n, then Ψ^{n+1} is the space of all oriented spheres in E^n.

Two spheres in E^n with centers $(a_1, \ldots, a_n)$, $(a_1', \ldots, a_n')$ and radii r, r' respectively are tangent, orientations taken into account, if

$$(a_1 - a_1')^2 + \ldots + (a_n - a_n')^2 = (r - r')^2 .$$

Use $a_1^2 + \ldots + a_n^2 = r^2 + C$ for both spheres to obtain the condition for tangency as

$$2a_1 a_1' + \ldots + 2a_n a_n' - 2rr' - C - C' = 0$$

or, in homogeneous coördinates,

$$2\alpha_1 \alpha_1' + \ldots + 2\alpha_n \alpha_n' - 2\lambda\lambda' - \mu\nu' - \nu\mu' = 0 .$$

Hence, two spheres of E^n are tangent when their corresponding points in Ψ^{n+1} are conjugate, that is, the line joining these points lies entirely in Ψ^{n+1} [6, §25].

4.2. A pencil of mutually tangent spheres in E^n corresponds to a line in Ψ^{n+1}. This pencil of spheres determines an "oriented complex co-direction" in E^n since it contains a point sphere and an incident oriented hyperplane. Corresponding to the hyperplane

$$x_n' - x_n = p_1 (x_1' - x_1) + \ldots + p_{n-1} (x_{n-1}' - x_{n-1})$$

at the point $(x_1, \ldots, x_n)$ is the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ \hline -\sqrt{pp+1} \\ 2(x_n - px) \\ 0 \end{bmatrix}$$

of Ψ^{n+1}, where

$$xx = \sum_{i=1}^{n} x_i^2, \quad px = \sum_{i=1}^{n-1} p_i x_i, \quad pp = \sum_{i=1}^{n-1} p_i^2;$$

this is the pencil of spheres

$$\sum_{i=1}^{n-1} (x_i' - x_i + tp_i)^2 + (x_n' - x_n - t)^2 = t^2 \left(\sum_{i=1}^{n-1} p_i^2 + 1 \right)$$

passing through $(x_1, \ldots, x_n)$ and having their centers on the line normal to the hyperplane at this point.

For later calculations it will be convenient to replace $-p, \ldots, -p_{n-1}, 1$ by homogeneous $u_1, \ldots, u_{n-1}, u_n$. The line in Ψ^{n+1} corresponding to the hyperplane

$$u_1 (x_1' - x_1) + \ldots + u_n (x_n' - x_n) = 0$$

at the point $(x_1, \ldots, x_n)$ is then

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hdashline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hdashline 0 \\ xx \\ 1 \end{bmatrix} + t \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hdashline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix},$$

where

$$xx = \sum_{i=1}^{n} x_i^2, \quad ux = \sum_{i=1}^{n} u_i x_i, \quad uu = \sum_{i=1}^{n} u_i^2.$$

Any convenient condition may be imposed on uu.

4.3. The contact structure on the $(2n-1)$-dimensional space of lines in Ψ^{n+1}, that is, the space of oriented co-directions in complex Euclidean space E^n, is obtained when the construction of 2.10. is carried out for the simple complex Lie algebra of type B_l or D_l, $l \geqslant 2$ and $l \geqslant 3$ respectively. However, it will be simpler to identify quantities geometrically if we proceed by using the description of 2.7, since now the groups are determined first.

Let

$$A = \left[\begin{array}{c|ccc} 2\cdot 1_n & & 0 & \\ \hline & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & -1 & 0 \end{array}\right]$$

be the matrix of the quadratic form defining Ψ^{n+1} in P^{n+1}. $SO\,(A\,;\,\mathbf{C})$, the special orthogonal group of this form, consists of matrices g in $SL\,(n+3;\,\mathbf{C})$ for which ${}^t gAg = A$. The connected centerless simple group $G = PSO\,(A;\,\mathbf{C}) = SO\,(A;\,\mathbf{C})/\{\text{center}\}$ is transitive on the lines of Ψ^{n+1} by Witt's theorem. Let l_0 be the line

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \\ \hline \lambda \\ \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline -1 \\ 0 \\ 0 \end{bmatrix}$$

of Ψ^{n+1}, joining

$${}^t(0, \dots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \dots, 0, 1 \mid -1, 0, 0)\,;$$

this corresponds to the pencil of spheres

$$\sum_{i=1}^{n-1} x_i'^2 + (x_n' - t)^2 = t^2$$

tangent to the hyperplane $x_n = 0$ at the origin of E^n, suitably oriented, as in 4.2. Let P denote the isotropy subgroup of l_0. Then

$$
\begin{aligned}
G/P &= \text{space of lines in } \Psi^{n+1} \\
&= \text{space of pencils of mutually tangent oriented spheres in } E^n \\
&= \text{space of oriented co-directions in complex } E^n.
\end{aligned}
$$

The Lie algebra $\mathfrak{g}$ of G consists of $(n+3)$ by $(n+3)$ matrices X for which ${}^tXA + AX = 0$. The matrices of $\mathfrak{g}$ are of the form

$$
\left[\begin{array}{ccc|ccc}
 & & & b_1 & c_1 & d_1 \\
 & n \text{ by } n \text{ skew-} & & \vdots & \vdots & \vdots \\
 & \text{symmetric} & & b_{n-1} & c_{n-1} & d_{n-1} \\
 & & & b_n & c_n & d_n \\
\hline
b_1 \ldots & b_{n-1} & b_n & 0 & c & d \\
2d_1 \ldots & 2d_{n-1} & 2d_n & -2d & e & 0 \\
2c_1 \ldots & 2c_{n-1} & 2c_n & -2c & 0 & -e
\end{array}\right].
$$

P consists of those elements of G which send the subspace of $\mathbf{C}^{n+3}$ spanned by

$$
{}^t(0, \ldots, 0, 0 \mid 0, 0, 1) \quad \text{and} \quad {}^t(0, \ldots, 0, 1 \mid -1, 0, 0)
$$

into itself; the Lie algebra $\mathfrak{p}$ of P consists of those elements of $\mathfrak{g}$ which do the same. Hence, the matrices of $\mathfrak{p}$ are of the form

$$
\left[\begin{array}{ccc|ccc}
 & & b_1 & b_1 & c_1 & 0 \\
 & (n-1) \text{ by } (n-1) & \vdots & \vdots & \vdots & \vdots \\
 & \text{skew-symmetric} & b_{n-1} & b_{n-1} & c_{n-1} & 0 \\
-b_1 \ldots & -b_{n-1} & 0 & b_n & c_n & -d \\
\hline
b_1 \ldots & b_{n-1} & b_n & 0 & c & d \\
0 \ldots & 0 & -2d & -2d & e & 0 \\
2c_1 \ldots & 2c_{n-1} & 2c_n & -2c & 0 & -e
\end{array}\right].
$$

Note that $\mathfrak{g}$ and $\mathfrak{p}$ have dimensions $\frac{1}{2}(n+3)(n+2)$ and $\frac{1}{2}(n-1)(n-2) + 2n + 3 = \frac{1}{2}(n+3)(n+2) - 2n + 1$, respectively, in agreement with G/P having dimension $2n-1$.

4.4. For $n \geqslant 2$, set $n + 3 = 2l + 1$ or $2l$ according as n is even or odd. $\mathfrak{g}$ is of type B_l or D_l, $l \geqslant 2$ and $l \geqslant 3$ respectively.

For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take matrices of the form

$$H = \operatorname{diag}\left[\,0 \;\middle|\; \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & h_{l-2} \\ -h_{l-2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_{l-1} & 0 & 0 \\ h_{l-1} & 0 & 0 & 0 \\ 0 & 0 & h_l & 0 \\ 0 & 0 & 0 & -h_l \end{bmatrix}\right];$$

the first row and column occur only in case B_l, it is suppressed for case D_l. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = (n+1)\operatorname{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \frac{1}{2}\operatorname{tr}(XY)$ for convenience.

Let W in $\mathfrak{p}$ be

$$W = \left[\begin{array}{c|cc|ccc}
0 & & 0 & & 0 & \\ \hline
0 & & 0 & & 0 & \\
 & & 0 & 0 & -\frac{1}{2} & 0 \\ \hline
 & & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & & & & \\
 & & 0 & 0 & 0 & 0 \\
 & & -1 & -1 & 0 & 0
\end{array}\right]$$

For H in $\mathfrak{h}$ we have $[H, W] = -(h_{l-1}+h_l)\,W$, so $\rho(H) = -(h_{l-1}+h_l)$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $W = E_\rho$ is the corresponding root vector.

For X in $\mathfrak{g}$ as described in 4.3, direct calculation shows $[X, W] = 0$ implies X is in $\mathfrak{p}$ and $b_n + e = 0$; thus the centralizer of W in $\mathfrak{g}$ consists of those elements of $\mathfrak{p}$ with $b_n + e = 0$. For X in $\mathfrak{p}$ now, the same calculation gives $[X, W] = -(b_n+e)\,W$, so $[X, W] = \rho(X)\,W$ with ρ extended to $\mathfrak{p}$ by $\rho(X) = -(b_n+e)$. Finally, W is orthogonal to $\mathfrak{p}$ with respect to the Killing form. Hence, (a', c', b') of 2.7 are satisfied, and W is the element of $\mathfrak{g}$ giving the contact structure on G/P.

The origin of the element W is not immediately evident. It was obtained by determining the maximal root and corresponding root vector for Lie algebras of type B_l and D_l when the quadratic form is

$$\xi_0^2 + 2\xi_1\xi_{l+1} + \dots + 2\xi_l\xi_{2l}$$

and then passing to the form

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu\nu$$

by conjugating by the element of $PSL(n+3; \mathbf{C})$ which corresponds to the "line-sphere transformation". This will be described further in the next section.

4.5. Let $\mathfrak{m}$ be the $(2n-1)$-dimensional supplement to $\mathfrak{p}$ in $\mathfrak{g}$ consisting of matrices of the form

$$\left[\begin{array}{ccc|c|ccc}
 & & & -b_1 & b_1 & 0 & d_1 \\
 & 0 & & \vdots & \vdots & \vdots & \vdots \\
 & & & -b_{n-1} & b_{n-1} & 0 & d_{n-1} \\
b_1 & \dots & b_{n-1} & 0 & 0 & 0 & d_n \\
\hline
b_1 & \dots & b_{n-1} & 0 & 0 & 0 & d_n \\
2d_1 & \dots & 2d_{n-1} & 2d_n & -2d_n & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

cf. 2.12. For X in $\mathfrak{m}$ we have

$$X^2 = \left[\begin{array}{cc|ccc} 0 & & & 0 & \\ & -bb & bb & 0 & bd \\ \hline & -bb & bb & 0 & bd \\ 0 & -2bd & 2bd & 0 & 2dd \\ & 0 & 0 & 0 & 0 \end{array}\right].$$

where

$$bb = \sum_{i=1}^{n-1} b_i^2, \; bd = \sum_{i=1}^{n-1} b_i d_i, \; dd = \sum_{i=1}^{n-1} d_i^2 .$$

The product of any three matrices of $\mathfrak{m}$ is zero. Especially,

$$\exp X = 1_{n+3} + X + \frac{1}{2} X^2 .$$

In order to establish classically identifiable coordinates on G/P as in 2.12, we must determine X in $\mathfrak{m}$ so that $(\exp X) \cdot l_0$ is the line of Ψ^{n+1} described in 4.2. With X in $\mathfrak{m}$ as above, $(\exp X) \cdot l_0$ is the line joining the points

$$(\exp X) \cdot \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} d_1 \\ \vdots \\ d_{n-1} \\ d_n + \frac{1}{2} bd \\ \hline d_n + \frac{1}{2} bd \\ dd \\ 1 \end{array}\right]$$

and

$$
(\exp X)\cdot\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hdashline -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n-2bd \\ 0 \end{bmatrix}.
$$

On this line we can identify the point sphere when $\lambda = 0$, giving

$$
\begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n+\frac{1}{2}bd \\ \hdashline d_n+\frac{1}{2}bd \\ dd \\ 1 \end{bmatrix} + \frac{d_n+\frac{1}{2}bd}{1+bb}\begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n-2bd \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hdashline 0 \\ xx \\ 1 \end{bmatrix},
$$

and the incident oriented hyperplane when $v = 0$, giving

$$\begin{bmatrix} -2b_1 \\ \vdots \\ -2b_{n-1} \\ 1-bb \\ \hdashline -1-bb \\ 4d_n - 2bd \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \hdashline -\sqrt{uu} \\ 2ux \\ 0 \end{bmatrix}.$$

These equations will be satisfied if be we impose the condition $\sqrt{uu} = 1 + bb$ on uu, or

$$u_i = -2b_i, \quad u_n = 1 - bb,$$

$i = 1, 2, \ldots, n-1$, and then set

$$b_i = -\frac{1}{2} u_i,$$

$$d_i = x_i - \frac{1}{2} u_i x_n, \qquad\qquad i = 1, 2, \ldots, n-1$$

$$d_n = \frac{1}{4} \sum_{i=1}^{n-1} u_i x_i + \frac{1}{2} x_n .$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, \ldots, x_n, u_1, \ldots, u_n$ on G/P as in 2.12 and 4.2.

4.6. From 2.12, the form ω on G/P is obtained as

$$\omega = \langle W, (\exp X)^{-1} d(\exp X) \rangle$$

with
$$(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2}[X, dX].$$

Take X as in 4.5 and let the entries of dX be denoted as those of X with primes affixed. Then

$$(\exp X)^{-1}\, d\,(\exp X) \;=\; \left[\begin{array}{cc|ccc}
 & -b_1' & b_1' & 0 & d_1' \\
0 & \vdots & \vdots & \vdots & \vdots \\
 & -b'_{n-1} & b'_{n-1} & 0 & d'_{n-1} \\
b_1' \ldots b'_{n-1} & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\ \hline
b_1' \ldots b'_{n-1} & 0 & 0 & 0 & d_n' - \frac{1}{2}c \\
2d_1' \ldots 2d'_{n-1} & 2d'_n - c & -2d_n' + c & 0 & 0 \\
0 \ldots 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

where

$$c \;=\; \sum_{i=1}^{n-1} (b_i d_i' - d_i b_i'),$$

and consequently, from the definition of W in 4.4, $\omega = c - 2d_n'$. Using the expressions in 4.5 for $b_1, \ldots, b_{n-1}, d_1, \ldots, d_n$ in terms of $x_1, \ldots, x_n, u_1, \ldots, u_n$, we obtain

$$\omega \;=\; -\sum_{i=1}^{n-1} u_i dx_i \;-\; \left[1 - \frac{1}{4}\sum_{i=1}^{n-1} u_i^2\right] dx_n$$

or, since $1 - \dfrac{1}{4}\displaystyle\sum_{i=1}^{n-1} u_i^2 = u_n$,

$$\omega \;=\; -\,(u_1\, dx_1 + \ldots + u_n\, dx_n)\,.$$

This identifies the contact structure with the classical one as in 2.12 and 4.2.

4.7. The real contact structure on the $(2n-1)$-dimensional space of oriented co-direction in real Euclidean space E^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the two-component centerless group $PSO\,(A;\,\mathbf{R})$ consisting of real contact automorphisms.