Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 25 (1979)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON LIE'S HIGHER SPHERE GEOMETRY
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Kapitel: 3. CO-DIRECTIONS IN PROJECTIVE SPACE
DOI: https://doi.org/10.5169/seals-50373

is one-to-one on an open neighborhood U of x_0 in G/P and $(\exp X) \cdot x_0$ is identifiable as the point $(x_1, ..., x_n)$ and the incident hyperplane

$$x_n' - x_n = p_1 (x_1' - x_1) + ... + p_{n-1} (x_{n-1}' - x_{n-1}) .$$

Now, $(\exp X) \cdot x_0 \to (\exp X) \cdot b_0$ is a section of the bundle G/P_1 over U and, via this section, the form ω on G/P_1 pulls down to

$$\omega_0 ((\exp X)^{-1} d (\exp X))$$

which, when expressed in terms of $x_1, ..., x_n, p_1, ..., p_{n-1}$, will be identified with

$$dx_n - p_1 dx_1 - ... - p_{n-1} dx_{n-1}$$

up to a constant multiple $a \neq 0$. For this latter calculation we will use

$$(\exp X)^{-1} d (\exp X) = \frac{1 - e^{-ad\, X}}{ad\, X} (dX)$$

$$= dX - \frac{1}{2} [X, dX] + \frac{1}{6} [X, [X, dX]] - ...$$

[4, (10.2)], a series which is finite since $\mathfrak{m}$ is nilpotent. In fact, our choice of X will make the series for $\exp X$ themselves finite. The constant $a \neq 0$ could be made unity by using instead the section $(\exp X) \cdot x_0 \to (\exp X) g^{-1} \cdot b_0$, where g in P is chosen so that $\chi (g) = a$. This amounts to following the original section by R_a^{-1} in the bundle.

3. Co-directions in projective space

The contact structure on the $(2n-1)$-dimensional space of co-directions in complex projective space P^n, described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type A_n, $n \geqslant 1$.

3.1 Let $\mathfrak{g} = \mathfrak{sl} (n + 1;\ \mathbf{C})$, complex $(n+1)$ by $(n+1)$ matrices of trace zero. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take the diagonal matrices of $\mathfrak{g}$. Let δ_i, $i = 0,1, ..., n$ be the linear function on $\mathfrak{h}$ which assigns to $H = \text{diag}\, (h_1, ..., h_n)$ in $\mathfrak{h}$ the i^{th} diagonal element: $\delta_i (H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\delta_i - \delta_j \quad i, j = 0, 1, ..., n$$
$$\text{and } i \neq j$$

and the root vector E_α corresponding to the root α is

$$E_{\delta_i-\delta_j} = E_{ij},$$

the matrix with 1 in the i^{th} row and j^{th} column and 0s elsewhere [4, (16.2)]. A system of simple roots is

$$\delta_0 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n,$$

for which the maximal root is

$$\rho = (\delta_0 - \delta_1) + (\delta_1 - \delta_2) + \ldots + (\delta_{n-1} - \delta_n) = \delta_0 - \delta_n$$

[4, App., Table E]. The Killing form of $\mathfrak{g}$ is $\langle X, Y \rangle = 2(n+1) \operatorname{tr}(XY)$, but we replace this with $\langle X, Y \rangle = \operatorname{tr}(XY)$ for convenience. Then the H_α in $\mathfrak{h}$ are given by

$$H_{\delta_i-\delta_j} = \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$$

with 1 and -1 in the i^{th} and j^{th} entry, respectively. Especially,

$$H_\rho = \operatorname{diag}(1, 0, \ldots, 0, -1).$$

We have

$$\langle H_\rho, H_{\delta_i-\delta_j} \rangle \begin{cases} < 0 & j = 0 \text{ or } i = n \\ \geqslant 0 & \text{otherwise} \end{cases},$$

so that $\mathfrak{p}$ in (i) of 2.9 consists of matrices of the form

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & * & \cdots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

of trace zero, where the starred entries are arbitrary.

3.2 The connected centerless simple group $G = PSL(n+1; \mathbf{C}) = SL(n+1; \mathbf{C})/\{\text{center}\}$ is transitive on the space consisting of points x and incident hyperplanes u, $ux = 0$, in P^n, as in 2.5. The isotropy subgroup P of the incident point and hyperplane

$$x_0 = {}^t(1, 0, \ldots, 0), \quad u_0 = (0, \ldots, 0, 1)$$

has exactly $\mathfrak{p}$ for its Lie algebra. Hence, the homogeneous contact manifold which the construction of 2.10 gives is

$$\begin{aligned} G/P &= \text{space of incident points and hyperplanes in } P^n \\ &= \text{space of co-directions in complex } P^n. \end{aligned}$$

3.3 Let $\mathfrak{m}$ be the $(2n-1)$-dimensional supplement to $\mathfrak{p}$ in $\mathfrak{g}$ consisting of matrices of the form

$$\begin{bmatrix} 0 & & & \\ * & & 0 & \\ \vdots & & & \\ * & \cdots & * & 0 \end{bmatrix};$$

cf. 2.12. The product of any two matrices of $\mathfrak{m}$ has a nonzero entry only in the n^{th} row and 0^{th} column; the product of any three is zero. Set

$$X = \begin{bmatrix} 0 & & & \\ x_1 & & 0 & \\ \vdots & & & \\ x_{n-1} & & & \\ x_n - \frac{1}{2}\sum p_i x_i & p_1 \cdots p_{n-1} & & 0 \end{bmatrix},$$

where the summation is over $i = 1, 2, \ldots, n-1$. X is in $\mathfrak{m}$ and

$$\exp X = 1_{n+1} + X + \frac{1}{2}X^2 = \begin{bmatrix} 1 & & & \\ x_1 & \ddots & & 0 \\ \vdots & & \ddots & \\ x_{n-1} & 0 & & \ddots \\ x_n & p_1 \cdots p_{n-1} & & 1 \end{bmatrix},$$

$$(\exp X)^{-1} = 1_{n+1} - X + \frac{1}{2}X^2 = \begin{bmatrix} 1 & & & & \\ -x_1 & \ddots & & 0 & \\ \vdots & & \ddots & & \\ -x_{n-1} & & 0 & \ddots & \\ -x_n + \sum p_i x_i & -p_1 & \dots & -p_{n-1} & 1 \end{bmatrix}.$$

The point

$$x = (\exp X)\cdot x_0 = {}^t(1, x_1, \dots, x_n)$$

is incident with the hyperplane

$$u = u_0 \cdot (\exp X)^{-1} = (-x_n + \sum p_i x_i, -p_1, \dots, -p_{n-1}, 1),$$

and the hyperplane $ux' = 0$, $x' = {}^t(1, x_1', \dots, x_n')$, is

$$x_n' - x_n = p_1(x_1' - x_1) + \dots + p_{n-1}(x_{n-1}' - x_{n-1}).$$

Thus, this choice of X establishes the classically identifiable coordinates $x_1, \dots, x_n, p_1, \dots, p_{n-1}$ on G/P.

3.8 From $\rho = \delta_0 - \delta_n$, we have $W = E_\varrho = E_{0n}$ in (iii) of 2.9 and $\omega_0(X) = \langle W, X \rangle$ is the $n0$-entry of X. The form ω on G/P is obtained as $\omega = \omega_0((\exp X)^{-1} d(\exp X))$ with

$$(\exp X)^{-1} d(\exp X) = dX - \frac{1}{2}[X, dX]$$

as in 2.12. For X as in 3.3, the only nonzero entry in $[X, dX]$ is the $n0^{th}$ and it is $\sum p_i dx_i = -\sum x_i dp_i$. Hence

$$(\exp X)^{-1} d(\exp X) = \begin{bmatrix} 0 & & & \\ dx_1 & & 0 & \\ \vdots & & & \\ dx_{n-1} & & & \\ dx_n - \sum p_i dx_i & dp_1 \dots dp_{n-1} & & 0 \end{bmatrix},$$

and the $n0$-entry is

$$\omega = dx_n - p_1 dx_1 - \dots - p_{n-1} dx_{n-1} .$$

This identifies the contact structure with the classical one as in 2.12.

3.5 The real contact structure on the $(2n-1)$-dimensional space of co-directions in real projective space P^n is described by viewing all quantities in the foregoing discussion as being real. Especially, G_0 of 2.11 is the connected centerless group $PSL(n+1; \mathbf{R})$ consisting of real contact automorphisms.

4. Higher sphere geometry

4.1 In complex Euclidean space E^n, the equation

$$x_1'^2 + \dots + x_n'^2 - 2a_1 x_1' - \dots - 2a_n x_n' + C = 0$$

describes a sphere with center $(a_1, \dots, a_n)$ and complex radius r given by

$$r^2 = a_1^2 + \dots + a_n^2 - C .$$

When $r \neq 0$, the two choices of sign for r is said to give two "orientations" to the sphere. Thus, the $n+2$ coordinates $a_1, \dots, a_n, r, C$, which are related by

$$a_1^2 + \dots + a_n^2 - r^2 - C = 0 ,$$

describe the space of oriented spheres in E^n [6, §25].

Introduce homogeneous coordinates by

$$a_i = \frac{\alpha_i}{\nu}, \quad r = \frac{\lambda}{\nu}, \quad C = \frac{\mu}{\nu},$$

$i = 1, 2, \dots, n$. Then the oriented spheres of E^n correspond to certain points of the quadric Ψ^{n+1} in P^{n+2} described by

$$\alpha_1^2 + \dots + \alpha_n^2 - \lambda^2 - \mu\nu = 0 .$$

The sphere corresponding to the point $(\alpha_1, \dots, \alpha_n, \lambda, \mu, \nu)$ of Ψ^{n+1} is

$$\nu(x_1'^2 + \dots + x_n'^2) - 2\alpha_1 x_1' - \dots - 2\alpha_n x_n' + \mu = 0 .$$

Ordinary spheres have finite nonzero radius r, so $\nu \neq 0$. For $\nu = 0$, we obtain oriented hyperplanes. For $\lambda = 0$, we obtain point spheres or hyperplanes with isotropic hyperplane coordinate vector; these carry no