## 2. Homogeneous contact manifolds

We formulate the notion of contact manifold in terms of complex analytic manifolds; the definitions apply equally to real smooth manifolds. Especially, if the complex analytic manifold is a smooth algebraic variety defined over **R** and if its various structures are defined over **R**, then the elementary assertions here apply to the set of real points of the variety.

Throughout this section we indicate proofs only when they differ from those of Boothby [1, 2] or Wolf [7, 8].

*2.1.* *A complex contact manifold* is a complex manifold $M$ of odd dimension $2n - 1$ together with a *complex contact structure* which is prescribed by a family $\{ (U_\alpha, \omega_\alpha) \}$ consisting of an open cover $\{ U_\alpha \}$ of $M$ and holomorphic Pfaffian forms $\omega_\alpha$ on $U_\alpha$ satisfying:

(i) $\omega_\alpha \wedge (d\omega_\alpha)^{n-1}$ does not vanish on $U_\alpha$, i.e., $\omega_\alpha$ is of maximal rank;

(ii) if $U_\alpha \cap U_\beta$ is not empty, then $\omega_\beta = f_{\beta\alpha} \omega_\alpha$ on $U_\alpha \cap U_\beta$ with $f_{\beta\alpha}$ holomorphic and non-vanishing; and

(iii) the family $\{ (U_\alpha, \omega_\alpha) \}$ is maximal with respect to (i) and (ii).

A holomorphic map $g: M \to M'$ between two contact manifolds is a *contact transformation* if $\{ (g^{-1}U'_\alpha, g^* \omega'_\alpha) \}$ is contained in $\{ (U_\alpha, \omega_\alpha) \}$. [1, §2].

*2.2.* Let $M$ be the space of hypersurface elements in complex Euclidean space $E^n$ whose hyperplanes meet the $x_n$-axis, that is, points $(x_1, ..., x_n)$ and incident hyperplanes

$$x'_n - x_n = p_1 (x'_1 - x_1) + ... + p_{n-1} (x'_{n-1} - x_{n-1}),$$

where primes denote running coordinates. The single set of coordinates $x_1, ..., x_n, p_1, ..., p_{n-1}$ on $M$ together with the Pfaffian form

$$\omega = dx_n - p_1 dx_1 - ... - p_{n-1} dx_{n-1}$$

suffices to define a contact structure on $M$ [6, §63]. This is *the* classical contact manifold to which we will relate all others.

*2.3.* The contact structure on a complex manifold $M$ has been formulated by S. Kobayashi in terms of a principal **C**\*-bundle over $M$ [1, §2

and 7, §2]. Let $\{\,(U_\alpha,\,\omega_\alpha)\,\}$ be a contact structure on $M$, so that $\omega_\beta = f_{\beta\alpha}\omega_\alpha$ on $U_\alpha \cap U_\beta$. Define the holomorphic principal $\mathbf{C}^*$-bundle $\pi\colon B \to M$ using the transition functions $f_{\beta\alpha}^{-1} = \dfrac{1}{f_{\beta\alpha}}$ on $U_\alpha \cap U_\beta$; $\pi^{-1}(U_\alpha)$ is $U_\alpha \times \mathbf{C}^*$ and, with coordinate $z_\alpha$ on $\mathbf{C}^*$, $z_\beta = f_{\beta\alpha}^{-1} z_\alpha$ on $U_\alpha \cap U_\beta$. $\omega_\alpha$ on $U_\alpha$ pulls back by $\pi^*$ to a Pfaffian form on $\pi^{-1}(U_\alpha)$, again denoted $\omega_\alpha$. On $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ we have $\omega_\beta = f_{\beta\alpha}\,\omega_\alpha$ and $z_\beta = f_{\beta\alpha}^{-1} z_\alpha$ so that $z_\alpha\omega_\alpha = z_\beta\omega_\beta$; the $z_\alpha\omega_\alpha$ hence define a holomorphic Pfaffian form $\omega$ on $B$. Let $R_a$, $a$ in $\mathbf{C}^*$, denote the right action of $\mathbf{C}^*$ on $B$. The Pfaffian form $\omega$ satisfies:

(a) $(d\omega)^n$ does not vanish on $B$;

(b) $\omega$ vanishes on vectors tangent to the fibers of $B$; and

(c) $R_a^*\omega = a\omega$, $a$ in $\mathbf{C}^*$ [1, (2.1)].

Conversely, a holomorphic principal $\mathbf{C}^*$-bundle $B$ over $M$ together with a holomorphic Pfaffian form $\omega$ satisfying $(a, b, c)$ defines a contact structure on $M$. The $\omega_\alpha$ on $U_\alpha$ are obtained by pulling down $\omega$ by sections of $B$ over $U_\alpha$.

Complex contact structures on $M$ correspond uniquely to principal $\mathbf{C}^*$-bundles $\pi\colon B \to M$ equipped with a Pfaffian form $\omega$ satisfying $(a, b, c)$, up to isomorphism [1, (2.1)]. Contact transformations $M \to M'$ are exactly those homomorphisms $g\colon B \to B'$, $\pi' \circ g = g \circ \pi$ and $R'_a \circ g = g \circ R_a$, satisfying $g^*\omega' = \omega$. Consequently, contact automorphisms of $M$ correspond to bundle automorphisms $g$ of $B$ which preserve $\omega$: $g^*\omega = \omega$ [1, (3.1)].

In case $M$ is *compact*, its group of all contact automorphisms is a *complex Lie group* which acts holomorphically on $B$ [1, (3.2) and 2, §1].

2.4.  Let $V$ be a complex manifold of dimension $n$ and $M$ the bundle of complex co-directions of $V$, that is, $M$ is obtained from the bundle $B$ which is the cotangent bundle of $V$ less its zero section by passing to the projective space of each fiber. $B$ is a principal $\mathbf{C}^*$-bundle over $M$. If $x_1, \ldots, x_n$ are coordinates on an open set $U$ of $V$, $\xi$ in $B$ may be written over $U$ as

$$\xi = u_1(\xi)\,dx_1 + \ldots + u_n(\xi)\,dx_n,$$

where the functions $u_i(\xi)$ are homogeneous of degree one; the functions $x_1, \ldots, x_n, u_1, \ldots, u_n$ define coordinates in $B$ over $U$. The Pfaffian form

$$\omega = u_1\,dx_1 + \ldots + u_n\,dx_n$$

on $B$ satisfies $(a, b, c)$ of 2.3 and hence defines a contact structure on $M$. Again, this is classical [6, §63, p. 242]. We may cover $M$ over $U$ by open sets where some $u_i$ is not zero, say $u_n \neq 0$, and then set

$$u_1 = - p_1, \ ..., \ u_{n-1} = - p_{n-1}, \ u_n = 1;$$

this gives a section of $B$ over this open set and $\omega$ pulls down to

$$dx_n - p_1 dx_1 - ... - p_{n-1} dx_{n-1},$$

the description in 2.2.

2.5.   Let $V$ of 2.4 be complex projective space $P^n$. Points of $P^n$ are described by homogeneous coordinates $x_0, ..., x_n$, written as a column vector $x = {}^t (x_1, ..., x_n)$, and hyperplanes

$$ux' = u_0 x_0' + ... + u_n x_n' = 0$$

of $P^n$ by homogeneous coordinates $u_0, ..., u_n$, written as a row vector $u = (u_0, ..., u_n)$. A cotangent vector at $x$ is determined by the equation of a hyperplane $u$ incident with $x$, $ux = 0$; if $x$ is replaced by $\lambda x$, $u$ must be replaced by $u\lambda^{-1}$. Thus $B$ may be described by $(x, u)$, $ux = 0$, with $(\lambda x, u\lambda^{-1})$ equivalent to $(x, u)$. $M$ is consequently described by incident points and hyperplanes $(x, u)$, $ux = 0$; now $(\lambda x, u\mu^{-1})$ is equivalent to $(x, u)$. The Pfaffian form

$$\omega = udx = u_0 dx_0 + ... + u_n dx_n$$

is well-defined on $B$ and gives the contact structure on the space $M$ of co-directions, that is, on the space of hypersurface elements, of $P^n$ [6, §63, p. 242].

A projectivity of $P^n$, a transformation in $PSL (n + 1; \mathbf{C})$, which will be represented by $x \to gx$ with $g$ in $SL (n + 1; \mathbf{C})$, transforms hyperplanes by $u \to ug^{-1}$ and cotangent vectors and co-directions by $(x, u) \to (gx, ug^{-1})$. Since

$$g^* \omega = g^* (udx) = ug^{-1} d (gx) = ug^{-1} gdx = udx = \omega,$$

projectivities are contact transformations of $M$. In addition, since $ux = 0$, $udx = - {}^t x {}^t du$ and, hence, classical projective duality $(x, u) \to ({}^t u, - {}^t x)$ preserves $\omega$ and is a contact transformation [6, §62].

2.6.   Let $M$ be a complex contact manifold which is *algebraic*; $M$ is a smooth algebraic variety and the contact structure is given by a bundle $B \to M$ and Pfaffian form $\omega$ on $B$ which are algebraic.

Assume further that $M$ is *connected, compact*, and *homogeneous* under a *linear algebraic group* $G$ of contact automorphisms. Since $M$ is connected, we may assume $G$ is *connected*. We may also assume that $G$ acts *effectively* on $M$: only the identity element of $G$ acts as the identity transformation on $M$.

Now $G$ is *semi-simple* [1, §4]. For the radical of $G$, a normal solvable subgroup, has a fixed point in compact $M$ [3, (10.4)] and since, $G$ acts effectively on $M$, this radical is trivial. Thus $M$ is exhibited as $G/P$, with $G$ connected and semi-simple, and $P$ the isotropy subgroup of a point $x_0$ in $M$. Since $G/P$ is compact, $P$ is a *parabolic* subgroup of $G$ [3, (11.2); 4, §68 ff.; and 8, §2]. $P$ is its own normalizer in $G$, so contains the center of $G$; since $G$ acts effectively on $M$, this center is trivial. $G$ is *centerless*.

Since $G$ is a linear algebraic group, we will throughout view the elements of it and its Lie algebra $\mathfrak{g}$ as matrices. Thus: For $g$ in $G$ and $X$ in $\mathfrak{g}$, $Ad\,(g)\,X = gXg^{-1}$, a product of matrices. Left-invariant Pfaffian forms on $G$ are given by $\omega_0\,(g^{-1}dg)$, where $\omega_0$ is a linear function on $\mathfrak{g}$, and $dg$ is the matrix of differentials of the entries of $g$. The action of $Ad\,(g)$ on left-invariant Pfaffian forms on $G$, i.e., $Ad\,(g)^*$, is then $({}^tAd(g)\omega_0)\,(X)$ $= \omega_0\,(gXg^{-1})$.

*2.7.* From 2.3, $G$ acts on the principal $\mathbf{C}^*$-bundle $B$ over $M = G/P$. Let $b_0$ in $B$ lie over the point $x_0$ in $M$ fixed by $P$. If $g$ is in $P$, then $gb_0$ lies over $x_0$, so $gb_0 = R_a b_0 = b_0 a$ for a unique $a = \chi\,(g)$ in $\mathbf{C}^*$. $\chi\colon P \to \mathbf{C}^*$ is a homomorphism. $\chi$ is either surjective or trivial, and in the former case $G$ is transitive on $B$ since it is then transitive on $M$ and on the fibers of $B$ over $M$. In fact, $\chi$ *is* surjective [2, §2]; the key lemma of Boothby's argument [2, p. 277] may be replaced by: The centralizer in $\mathfrak{g}$ of a nonzero element of $\mathfrak{g}$ is never a parabolic subalgebra. Thus $B$ is exhibited as $G/P_1$ with $P_1$, the kernel of $\chi$, a subgroup of $P$. The bundle $B \to M$ is $G/P_1 \to G/P$ with fiber $P/P_1 \simeq \mathbf{C}^*$.

By means of the map $G \to G/P_1$, pull the Pfaffian form $\omega$ on $B = G/P$, which defines the contact structure, up to a left-invariant form $\omega_0\,(g^{-1}dg)$ on $G$. This form is $Ad\,(P_1)$-*invariant*:

$$\omega_0\,(gXg^{-1}) = \omega_0\,(X)\,, \quad g \text{ in } P_1\,, \; X \text{ in } \mathfrak{g}\,.$$

Let $\mathfrak{p}$ and $\mathfrak{p}_1$ denote the Lie algebras of $P$ and $P_1$, respectively. Conditions $(a, b, c)$ of 2.3 become:

(a)  $(d\omega_0)^n \neq 0$;

(b)  $\omega_0(X) = 0$, $X$ in $\mathfrak{p}$; and

(c)  $\omega_0(g^{-1}Xg) = \chi(g)\,\omega_0(X)$, $g$ in $P$, $X$ in $\mathfrak{g}$;

where $d\omega_0(X, Y) = -\dfrac{1}{2}\,\omega_0([X, Y])$ [1, (5.1), (5.2), (5.3)].

Since $\mathfrak{g}$ is semi-simple, its Killing form is non-degenerate and we may write

$$\omega_0(X) = \langle\, W, X \,\rangle,\quad X \text{ in } \mathfrak{g},$$

for a unique $W$ in $\mathfrak{g}$. Conditions $(a, b, c)$ now become:

(a')  the centralizer of $W$ in $\mathfrak{g}$ is $\mathfrak{p}_1$;

(b')  $\langle\, W, X \,\rangle = 0$, $X$ in $\mathfrak{p}$; and

(c')  $[X, W] = \chi'(X)\,W$, $X$ in $\mathfrak{p}$;

where $\chi'$ is the derivative of $\chi$ at the identity of $P$ [1, (5.6)].

As a consequence of (c'), $\rho = \chi'$ restricted to a Cartan subalgebra contained in $\mathfrak{p}$ is a root of $\mathfrak{g}$; $E_\rho = W$ may be taken as the corresponding root vector. When the roots of $\mathfrak{g}$ are ordered, $\rho$ is a positive root and $\rho + \alpha$, $\alpha$ a positive root, is not a root [1, (6.2)]. Hence $\rho$ is the *maximal* root for this ordering and $G$ is *simple* [1, (6.3) and 4, (25.6)].

*2.8*  Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, and choose a system of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Designate a subset of the simple roots as free and call the remaining simple roots non-free. An arbitrary root is called *free* if it contains a free simple root as a summand, and *non-free* if all its summands are non-free simple roots. A free root is necessarily positive. Besides free and non-free roots there are only the negatives of free roots [4, (69.23)]. If $\mathfrak{g}_\alpha$ denotes the root space for the root $\alpha$, then

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \text{ non-free}} \mathfrak{g}_\alpha + \sum_{\alpha \text{ free}} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$, that is, it corresponds to a parabolic subgroup of any connected complex Lie group $G$ having $\mathfrak{g}$ as its Lie algebra. Now, any parabolic subalgebra of $\mathfrak{g}$ is $Ad(G)$-conjugate to a parabolic subalgebra given by the above construction. Thus, once $\mathfrak{h}$ and the system of simple roots are fixed, the subsets of the simple roots classify parabolic subalgebras up to conjugacy [8, §2].

*2.9* Continuing 2.7: We may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{p}$, choose a system of simple roots, and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8 [8, §2]. The free and non-free roots are completely determined by the maximal root $\rho$, in that

$$\langle H_\rho, H_\alpha \rangle > 0 \text{ for } \alpha \text{ free,}$$

$$\langle H_\rho, H_\alpha \rangle = 0 \text{ for } \alpha \text{ non-free,}$$

where $H_\alpha$ in $\mathfrak{h}$ is defined by $\langle H_\alpha, H \rangle = \alpha(H)$, $H$ in $\mathfrak{h}$ [1, (6.5)]. Consequently, we may describe $\mathfrak{p}$, $\mathfrak{p}_1$, and $\omega_0$ for an algebraic homogeneous contact manifold in terms of the maximal root $\rho$ by

(i) $\mathfrak{p} = \mathfrak{h} + \displaystyle\sum_{\langle H_\rho, H_\alpha \rangle \geq 0} \mathfrak{g}_\alpha$,

(ii) $\mathfrak{p}_1 = $ elements $X$ of $\mathfrak{p}$ orthogonal to $H_\rho$, $\langle H_\rho, X \rangle = 0$, and

(iii) $\omega_0(X) = \langle E_\rho, X \rangle$, $X$ in $\mathfrak{g}$,

[7, p. 1035]. Since $G$ is connected and centerless with Lie algebra $\mathfrak{g}$, the groups $G$, $P$, $P_1$ and the form $\omega$ are completely determined.

*2.10.* Conversely, begin with a simple complex Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots. Using the maximal root $\rho$, define $\mathfrak{p}$, $\mathfrak{p}_1$, and $\omega_0$ as in (i, ii, iii) of 2.9. Take for $G$ the adjoint group of $\mathfrak{g}$, which is connected, centerless, and simple, and for $P$ and $P_1$ the subgroups corresponding to $\mathfrak{p}$ and $\mathfrak{p}_1$. Then, $\omega_0(g^{-1} dg)$ is a left-invariant, $Ad(P_1)$-invariant Pfaffian form on $G$, and defines a form $\omega$ on $G/P_1$. The map $X \rightarrow \langle H_\rho, X \rangle$, $X$ in $\mathfrak{p}$, gives rise to a homomorphism $\chi: P \rightarrow \mathbf{C}^*$. The form $\omega$ on the principal $\mathbf{C}^*$-bundle $G/P_1$ over $G/P$ satisfies $(a, b, c)$ of 2.3 and hence defines a contact structure making $G/P$ a compact homogeneous algebraic contact manifold [1, Th. C and 7, p. 1035].

In this manner, Boothby established that there is exactly one compact homogeneous algebraic contact manifold, up to isomorphism, for each type $A_n$, ..., $G_2$ of simple complex Lie algebra [1, (7.1)]. For these manifolds, the group $G$ is the identity component of the group of all contact automorphisms [7, (2.5)]. Boothby's classification [1, 2] was obtained with the assumptions that the complex contact manifold was compact, simply connected, and homogeneous under a complex Lie group of contact transformations, and used H. C. Wang's theory of compact homogeneous complex manifolds rather than parabolic subgroups. We may conclude that these contact manifolds are algebraic.

*2.11.* Let $G$ be a semi-simple complex Lie group and $G_0$ a real form of $G$ : $G_0$ is the set of elements of $G$ fixed under a complex conjugation $g \to \bar{g}$. We use a bar to denote the conjugate of an object, and the terms *real* and *stable* refer to the conjugation.

Let $P$ be a parabolic subgroup of $G$, and $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{p}$ the Lie algebras of $G$, $G_0$, $P$. $\mathfrak{g}_0 \cap \mathfrak{p}$ contains a stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\bar{\mathfrak{h}} = \mathfrak{h}$ [8, (2.6)]. If $\alpha$ is a root of $\mathfrak{g}$, so is $\bar{\alpha}$; $\alpha$ is real if $\bar{\alpha} = \alpha$. Now, choose a system of simple roots and find the subset of free simple roots so that $\mathfrak{p}$ is given by the construction of 2.8. Set $P_0 = G_0 \cap P$; $G_0/P_0$ is a subset of $G/P$. Wolf has shown that the following are *equivalent* :

(i)   real dimension of $G_0/P_0$ = complex dimension of $G/P$,

(ii)   $G_0/P_0$ is closed in $G/P$,

(iii)   the set of free roots is stable,

(iv)   $\mathfrak{p}$ or $P$ is stable, and

(v)   the algebraic manifold $G/P$ is defined over $\mathbf{R}$ and $G_0/P_0$ is its set of real points,

[8, (3.6)]. When these conditions hold, $G_0/P_0$ is the unique closed orbit of $G_0$ on $G/P$ [8, (3.4)]. We call $G_0/P_0$ a *real form* of $G/P$.

Let $M$ be a compact, algebraic homogeneous contact manifold. Assume that $M$ and its contact structure are *defined over* $\mathbf{R}$, that is, the principal $C^*$-bundle $B \to M$ and the Pfaffian form $\omega$ on $B$ are defined over $\mathbf{R}$. Let $P$ be the isotropy subgroup of a real point $x_0$ of $M$ in the group $G$ of contact automorphisms. Then the complex conjugation on $M$ defines one on $G$, $\bar{g}x = \overline{g\bar{x}}$, under which $P$ is stable. Hence, we obtain a real form $G_0$ of $G$ so that the real points of $M$ are $G_0/P_0$, $P_0 = G_0 \cap P$. That $\omega$ is defined over $\mathbf{R}$ means $W = E_\rho$ lines in $\mathfrak{g}_0$, and the maximal root $\rho$ is *real*. This is consistent with the stability of the set of free roots, as $\langle H_\rho, H_{\bar{\alpha}} \rangle > 0$ when $\langle H_\rho, H_\alpha \rangle > 0$. Consequently, the real forms of $M$ correspond to the conjugations of $\mathfrak{g}$ for which $\rho$ is real.

*2.12.* The method by which the contact structure on $G/P$ will be exhibited, in the next sections, in classical form 2.2 is the following.
Let

$$\mathfrak{m} = \sum_{\langle H_\rho, H_\alpha \rangle < 0} \mathfrak{g}_\alpha \, ;$$

$\mathfrak{m}$ is supplementary to $\mathfrak{p}$ in $\mathfrak{g}$ and of dimension $2n - 1$. We will determine $X$ near 0 in $\mathfrak{m}$ as a function of $x_1, ..., x_n, p_1, ..., p_{n-1}$ so that $X \to (\exp X) \cdot x_0$

is one-to-one on an open neighborhood $U$ of $x_0$ in $G/P$ and $(\exp X) \cdot x_0$ is identifiable as the point $(x_1, ..., x_n)$ and the incident hyperplane

$$x_n' - x_n = p_1 (x_1' - x_1) + ... + p_{n-1} (x_{n-1}' - x_{n-1}).$$

Now, $(\exp X) \cdot x_0 \to (\exp X) \cdot b_0$ is a section of the bundle $G/P_1$ over $U$ and, via this section, the form $\omega$ on $G/P_1$ pulls down to

$$\omega_0 \left( (\exp X)^{-1} d (\exp X) \right)$$

which, when expressed in terms of $x_1, ..., x_n, p_1, ..., p_{n-1}$, will be identified with

$$dx_n - p_1 dx_1 - ... - p_{n-1} dx_{n-1}$$

up to a constant multiple $a \neq 0$. For this latter calculation we will use

$$(\exp X)^{-1} d (\exp X) = \frac{1 - e^{-ad\, X}}{ad\, X} (dX)$$

$$= dX - \frac{1}{2} [X, dX] + \frac{1}{6} [X, [X, dX]] - ...$$

[4, (10.2) ], a series which is finite since $\mathfrak{m}$ is nilpotent. In fact, our choice of $X$ will make the series for $\exp X$ themselves finite. The constant $a \neq 0$ could be made unity by using instead the section $(\exp X) \cdot x_0 \to (\exp X) g^{-1} \cdot b_0$, where $g$ in $P$ is chosen so that $\chi(g) = a$. This amounts to following the original section by $R_a^{-1}$ in the bundle.

## 3. Co-directions in projective space

The contact structure on the $(2n-1)$-dimensional space of co-directions in complex projective space $P^n$, described in 2.5, is obtained when the construction of 2.10 is carried out for the simple complex Lie algebra of type $A_n$, $n \geqslant 1$.

*3.1* Let $\mathfrak{g} = \mathfrak{sl}(n + 1; \mathbf{C})$, complex $(n+1)$ by $(n+1)$ matrices of trace zero. For Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ take the diagonal matrices of $\mathfrak{g}$. Let $\delta_i$, $i = 0, 1, ..., n$ be the linear function on $\mathfrak{h}$ which assigns to $H = \text{diag}\ (h_1, ..., h_n)$ in $\mathfrak{h}$ the $i^{th}$ diagonal element: $\delta_i (H) = h_i$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\delta_i - \delta_j \quad i, j = 0, 1, ..., n$$

$$\text{and } i \neq j$$